Performance Guarantee

The problem: $\boxed{\begin{array}{ll} \min \|x\|_0 & \\ \text{s.t. } \; Ax = b & \text{———} (P_0) \end{array}}$

We will study three algorithms:

(i) OMP

(ii) Thresholding

(iii) l_1-minimization (Basis Pursuit).

Goal: Provide <u>performance Guarantee</u>. will the algorithm give you the solution?

OMP Performance Guarantee

<u>Theorem</u> (4.3 Elad)

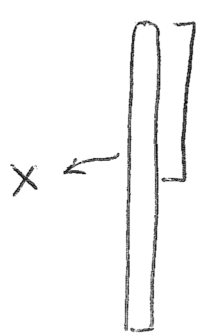
For a system $Ax=b$, $A \in \mathbb{R}^{n \times m}$ full rank, $n < m$, if a solution x exists and x obeys

$$\|x\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(A)}\right),$$

then OMP with stopping criteria $\|r^k\| \leq \epsilon_0$ $(\epsilon_0 = 0)$ is guaranteed to find x exactly.

<u>Proof</u>:

WLOG, assume the sparest solution is such that all its k_0 non-zeros are at the beginning.



k_0 non-zero. order according to $|x_i|$.

Then,

$$b = Ax = \sum_{t=1}^{k_0} x_t a_t .$$

sum of the first k_0 elements.

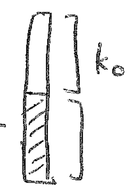
At the first OMP step, $r^k = r^0 = b$.

we assume A is column normalized.

$$\varepsilon(j) = \min_z \|a_j z - b\|^2 = \|b\|^2 - (a_j^T b)^2 \geq 0$$

Therefore, in order <u>NOT TO PICK</u> the wrong column, must have

$$|a_1^T b| > |a_i^T b|, \quad i > k_0$$



a_1 : just choose one of the k_0's.

Thus,

$$\left| a_1^T \left(\sum_{t=1}^{k_0} x_t a_t \right) \right| > \left| a_i^T \left(\sum_{t=1}^{k_0} x_t a_t \right) \right|$$

$$\Rightarrow \underbrace{\left| \sum_{t=1}^{k_0} x_t a_1^T a_t \right|} > \left| \sum_{t=1}^{k_0} x_t a_i^T a_t \right|$$

$$\left| \sum_{t=1}^{k_0} x_t a_1^T a_t \right| = \left| x_1 + \sum_{t=2}^{k_0} a_1^T a_t x_t \right|$$

$|a+b| > |a| - |b|$

$$\geq |x_1| - \left| \sum_{t=2}^{k_0} a_1^T a_t x_t \right|$$

$(a+b)^2 > |a|^2 + |b|^2 - 2|a||b|$

$$\geq |x_1| - \sum_{t=2}^{k_0} |x_t| \, |a_1^T a_t|$$

$\boxed{\left|\sum x\right| \leq \sum |x|}$

$$\geq |x_1| - \sum_{t=2}^{k_0} |x_t| \left(\max_{i,j} |a_i^T a_j| \right)$$

$$= |x_1| - \sum_{t=2}^{k_0} |x_t| \, \mu(A)$$

$$\geq |x_1| - \sum_{t=2}^{k_0} |x_1| \, \mu(A) = |x_1| \left(1 - \mu(A)(k_0 - 1)\right).$$

Now, the right hand side is

$$\left| \sum_{t=1}^{k_0} x_t a_i^T a_t \right| \leq \sum_{t=1}^{k_0} |x_t| |a_i^T a_t|$$

$$\leq \sum_{t=1}^{k_0} |x_t| \mu(A)$$

$$\leq |x_1| \mu(A) k_0 .$$

~~So if we want the correct support, then we need~~

This implies

$$|x_1| (1-\mu(A)) (k_0 - 1) > |x_1| \mu(A) k_0 .$$

$$\Rightarrow \quad 1 + \mu(A) > 2\mu(A) k_0$$

$$\Rightarrow \quad k_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(A)}\right) .$$

Therefore, if $k_0 < \frac{1}{2}\left(1+\frac{1}{\mu(A)}\right)$, then we are guaranteed to have $|a_1^T b| > |a_i^T b|$, and so the support is correct.

Once support ~~is~~ $S = \{j^*\}$ is found, OMP ensures that the same index will not be picked again. Also OMP ensures that the residue decreases.

Therefore, the algorithm is guaranteed to find the solution.

Thresholding Performance Guarantee

Theorem (4.4 Elad)

For a system $Ax=b$, $A \in \mathbb{R}^{n\times m}$, $n<m$, full rank, if a solution x exists, and obeys

$$\|x\|_0 < \frac{1}{2}\left(1+\frac{1}{\mu(A)}\frac{|x_{min}|}{|x_{max}|}\right),$$

then the thresholding algorithm with stopping criteria $\|r^k\| \le \epsilon_0$ ($\epsilon_0=0$) is guaranteed to find x exactly.

Interpretation:

$$\frac{1}{2}\left(1+\frac{1}{\mu(A)}\cdot\frac{|x_{min}|}{|x_{max}|}\right) < \frac{1}{2}\left(1+\frac{1}{\mu(A)}\right).$$

So ~~alg~~ although thresholding algorithm still works, it requires a stricter condition on the solution.

Proof: Successful thresholding requires this:

$$\underbrace{\min_{1\le i\le k_0} |a_i^T b|} > \max_{j>k_0} |a_j^T b|$$

$$\cancel{\Rightarrow} \min_{1\le i\le k_0} |a_i^T b| = \min_{1\le i\le k_0}\left|\sum_{t=1}^{k_0} x_t a_i^T a_t\right|$$

$$= \min_{1\le i\le k_0}\left| x_i + \sum_{\substack{t\ne i\\ 1\le t\le k_0}} x_t a_i^T a_t\right|$$

$$\ge \min_{1\le i\le k_0} |x_i| - \left|\sum_{t\ne i} x_t a_i^T a_t\right|$$

$$\ge \max_{1\le i\le k_0} |x_i| - \max_{1\le i\le k_0}\left|\sum_{t\ne i} x_t a_i^T a_t\right|$$

$$\ge \cancel{\max_{1\le i\le k_0}} |x_{min}| - (k_0-1)\mu(A)|x_{max}|$$

The right hand side is

$$\max_{j>k_0} |a_j^T b| = \max_{j>k_0} \left| \sum_{t=1}^{k_0} x_t a_j^T a_t \right|$$

$$\leq \max_{j>k_0} \sum_{t=1}^{k_0} |x_t| \, |a_j^T a_t|$$

$$\leq \sum_{t=1}^{k_0} |x_{max}| \, \mu(A)$$

$$= k_0 |x_{max}| \mu(A).$$

Therefore if we have

$$|x_{min}| - (k_0 - 1)\mu(A)|x_{max}| > k_0 |x_{max}| \mu(A),$$

then we will have successful thresholding.

$$\to \quad k_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(A)} \cdot \frac{|x_{min}|}{|x_{max}|}\right).$$

Basis Pursuit Guarantee

$$(P_1) \quad \min \|x\|_1 \quad \text{s.t.} \quad Ax = b.$$

How to solve this problem?

- Linear programming

Let $u \geq 0$, $v \geq 0$. Then let $x = u - v$. So

$$(P_1) \Rightarrow \min_{u,v} \ 1^T(u+v) \quad \text{s.t.} \quad [A \ \ -A]\begin{bmatrix} u \\ v \end{bmatrix} = b$$

LP format:
$$\min c^T x \quad \text{s.t.} \quad Ax = b, \quad Cx \leq d.$$

Theorem (4.5 Elad)

For a system $Ax = b$ ($A \in \mathbb{R}^{n\times m}$, $n<m$, A full rank).
If a solution x exists and obeys

$$\|x\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(A)}\right).$$

then the linear programming of (P_1) is guaranteed to find the solution exactly.

Proof Define this set

$$C = \left\{ y \;\middle|\; \begin{array}{l} \|y\|_1 \le \|x\|_1 \\ \|y\|_0 > \|x\|_0 \\ A(y-x) = 0 \end{array} \right\}.$$

solutions that have lower l_1 bound but is ~~more~~ Less sparse.

argument: If C is empty, then BP returns the most sparse solution.

Step 1: Let $e = y - x$. Then

① $\|y\|_1 \le \|x\|_1 \Rightarrow \|e+x\|_1 - \|x\|_1 \le 0$.

② $A(y-x) = 0 \Rightarrow Ae = 0$.

But $\|e+x\|_1 - \|x\|_1 \le 0$ can be further relaxed.
Assume the non-zeros of x are located in the beginning of the vector. Then,

$$\|e+x\|_1 - \|x\|_1 = \left(\sum_{j=1}^{k_0} |e_j + x_j| - |x_j|\right) + \underbrace{\left(\sum_{j>k_0} |e_j|\right)}_{\text{the remaining entry of } x \text{ are zero.}} \le 0$$

Since $|a+b| - |b| \ge -|a|$, we have

$$\sum_{j=1}^{k_0} |e_j + x_j| - |x_j| + \sum_{j>k_0} |e_j| \le 0$$

$$-\sum_{j=1}^{k_0} |e_j| + \sum_{j>k_0} |e_j| \le 0.$$

This can be written as

sum of the first k_0 entries

$$\|e\|_1 - 2\left(\mathbb{1}_{k_0}^T |e|\right) \leq 0$$

So $C \subseteq C_1 \overset{\text{def}}{=} \left\{ e \,\middle|\, e \neq 0,\ \|e\|_1 - 2\mathbb{1}_{k_0}^T |e| \leq 0,\ Ae = 0 \right\}$.

<u>Step 2</u>: $\quad Ae = 0 \implies A^T A e = 0$

$$\implies (A^T A - I)e = -e$$

So
$$|e| = |(A^T A - I)e| \leq |A^T A - I|\,|e| \leq \mu(A)(\mathbb{1} - I)|e|.$$

$$A^T A = \begin{pmatrix} 1 & a_1^T a_2 & \cdots & a_1^T a_n \\ \vdots & \ddots & & \vdots \\ a_n^T a_1 & \cdots & & 1 \end{pmatrix}$$

$$\implies |e| \leq \mu(A)\mathbb{1}|e| - \mu(A)|e|$$

$$\implies (1+\mu(A))|e| \leq \mu(A)\mathbb{1}|e|$$

$$\implies |e| \leq \frac{\mu(A)}{1+\mu(A)}\mathbb{1}|e|.$$

So $C \subseteq C_1 \subseteq C_2 \overset{\text{def}}{=} \left\{ e \,\middle|\, e \neq 0,\ \|e\|_1 - 2\mathbb{1}_{k_0}^T |e| \leq 0,\ |e| \leq \frac{\mu}{1+\mu}\mathbb{1}|e| \right\}$.

But C_2 is unbounded because $e \in C_2 \implies \alpha e \in C_2$.
So we should consider $\|e\|_1 = 1$.

Define $C_3 = \left\{ e \;\middle|\; \|e\|_1 = 1,\ 1 - 2\mathbb{1}_{k_0}^T |e| \le 0 \right.$

$\left. |e| \le \frac{\mu(A)}{1+\mu(A)} \underbrace{\mathbb{1}\,|e|} \right\}$

$= \mathbb{1}\, \mathbb{1}^T |e|$

$= \mathbb{1}\, \|e\|_1$

$= \mathbb{1}$.

Now, in order for e to satisfy

$$1 - 2\,\mathbb{1}_{k_0}^T |e| \le 0,$$

one has to concentrate its energy on the first k_0 entries. However $\|e\|_1 = 1$, and

$$|e_j| \le \frac{\mu(A)}{1+\mu(A)} \Rightarrow \text{the } k_0 \text{ entries have exactly } |e_j| = \frac{\mu}{1+\mu}.$$

$$\Rightarrow \quad 1 - 2\,\mathbb{1}_{k_0}^T |e| = 1 - 2k_0 \frac{\mu}{1+\mu} \le 0$$

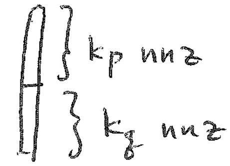
$$\Rightarrow \quad k_0 < \tfrac{1}{2}\left(1 + \frac{1}{\mu}\right).$$

Special Case: The two-ortho System

Theorem (4.1 Elad)

For a two-ortho system $[\Psi, \Phi]x = b$, if a solution x exists and it has k_p non-zeros in the first half, and k_q non-zeros in the second half, ~~then~~ and

$$\max(k_p, k_q) < \frac{1}{2\mu(A)},$$



then OMP with stopping criteria $\|r^k\| = 0$ will find the solution in exactly $k_0 = k_p + k_q$ steps

Proof: See textbook.

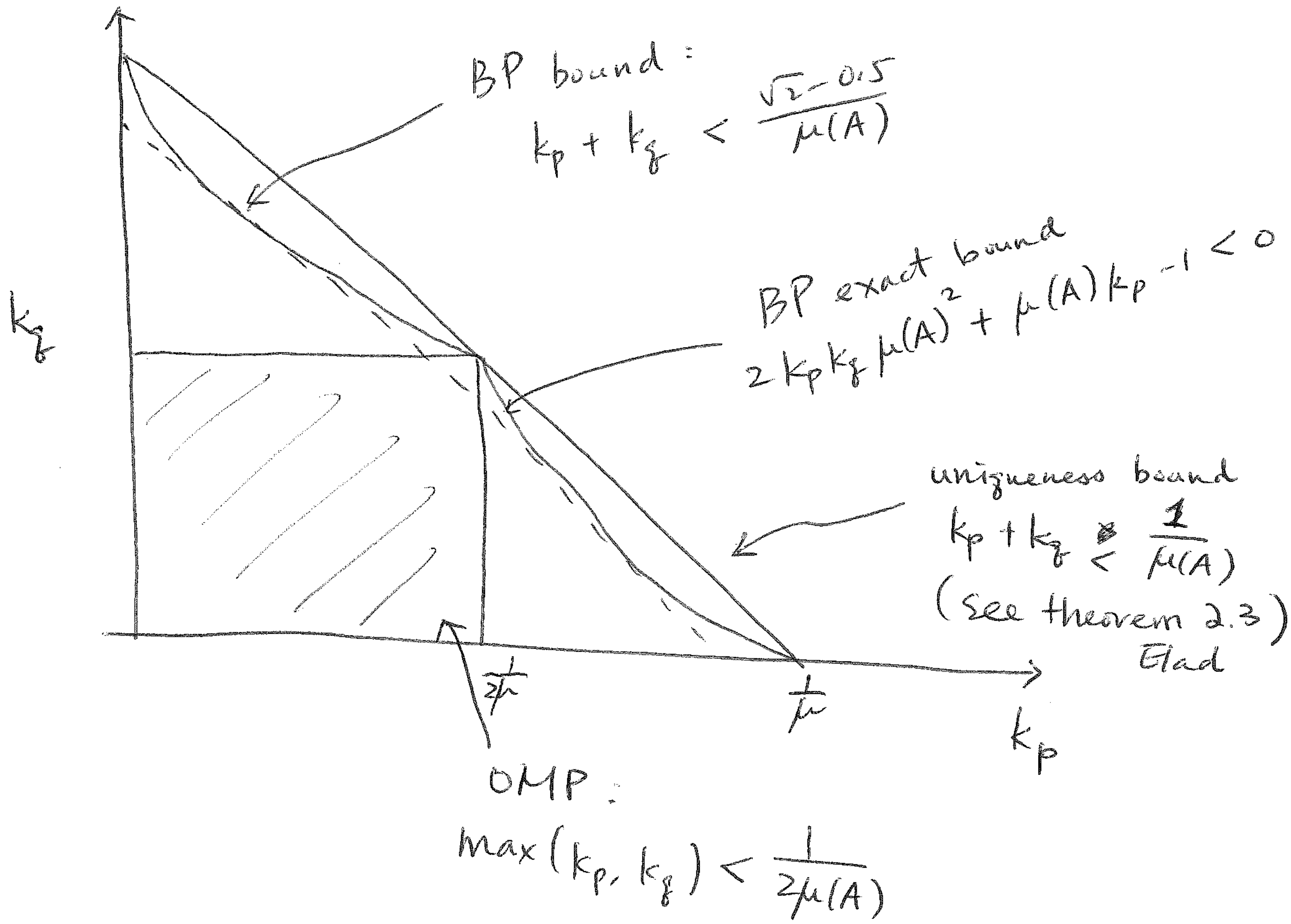
Theorem (4.2 Elad)

For a two ortho system $[\Psi, \Phi]x = b$, if a solution x exists and it has ~~k_q~~ k_p non-zeros in the first half and ~~k_q~~ $k_q \le k_p$ non-zeros in the second half, and

$$\|x\|_0 = k_p + k_q < \frac{\sqrt{2} - 0.5}{\mu(A)},$$

then the linear programming for (P_1) will be able to find that solution.

Proof: See textbook.



BP bound:
$k_p + k_q < \frac{\sqrt{2} - 0.5}{\mu(A)}$
BP exact bound
$2 k_p k_q \mu(A)^2 + \mu(A) k_p - 1 < 0.$
uniqueness bound
$k_p + k_q < \frac{1}{\mu(A)}$
(see theorem 2.3) Elad
k_q
k_p
$\frac{1}{2\mu}$
$\frac{1}{\mu}$
OMP:
$\max(k_p, k_q) < \frac{1}{2\mu(A)}$

Stability of Sparse Solution

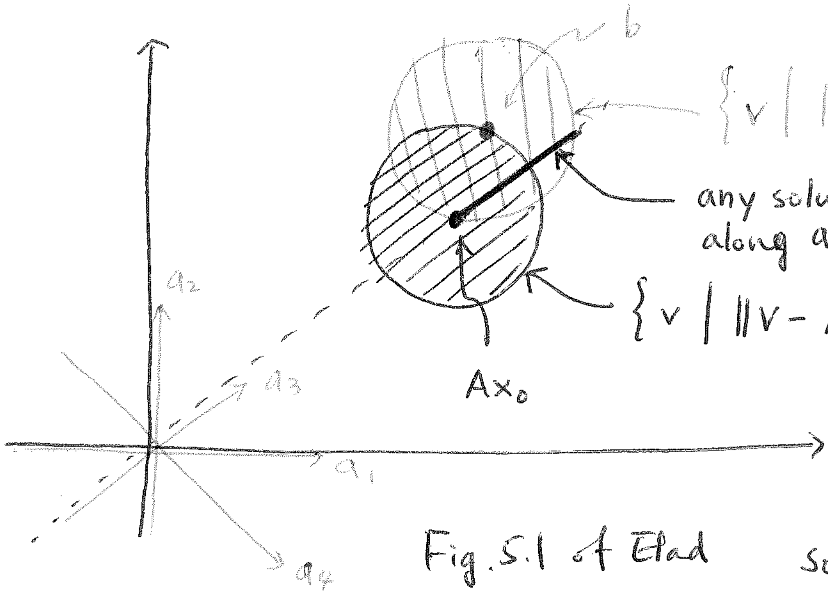
$$(P_0^\epsilon) \quad \min \|x\|_0 \quad \text{s.t. } \|Ax-b\| < \epsilon.$$

As ϵ increases, the feasible set enlarges, so there are more solutions to choose, and uniqueness will become a problem.

Example

$$A = \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \qquad x_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Generate $b = Ax_0 + e$, for some error vector e.



$\{v \mid \|v-b\| < \epsilon\}$

any solution on this line is along a_3, and $\|a_3 z - b\| < \epsilon$.

$\{v \mid \|v - Ax_0\| < \epsilon\}$

Fig. 5.1 of Elad

As ϵ increases, the support of the solution will also change. This will lead to unstable solutions. (See Fig 5.2 of Elad)

So instead of have a unique solution, now we have a segment of line that (i) has the lowest sparsity
(ii) is still feasible.

Good thing: support set remains the same.

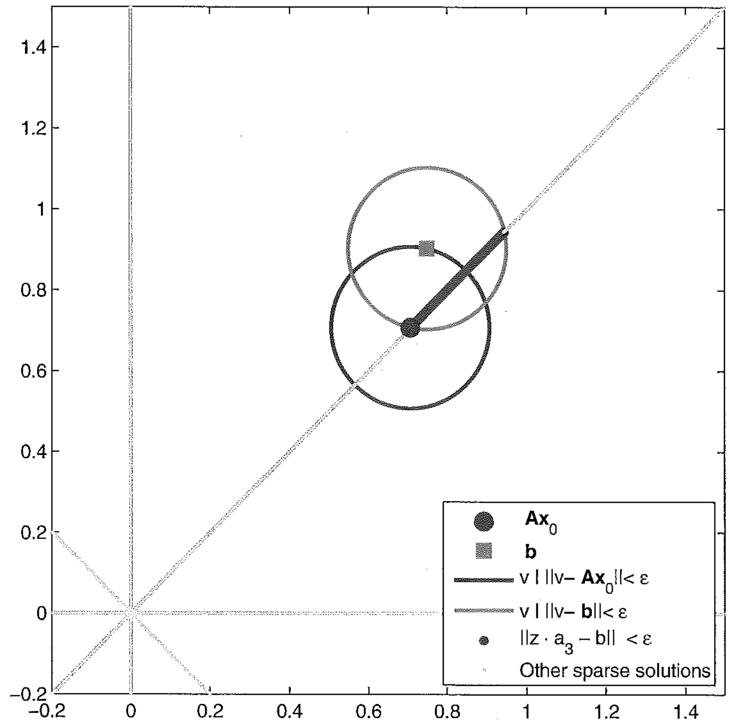


Fig. 5.1 A 2D demonstration of the lack of uniqueness for the noisy case, with a relatively weak noise.

solutions that are sparse? Figure 5.1 shows that solutions of the form $\mathbf{x}_0 = [0\;0\;x\;0]^T$ with some values of x are also feasible, while having the same cardinality.

Figure 5.2 presents the same experiment, this time with a stronger noise, $\epsilon = 0.6$. This leads to a different scenario, where not only we have lost uniqueness with respect to the same support, but other supports with cardinality $\|\mathbf{x}\|_0 = 1$ are possible, and in fact, even the null solution is included, implying that this is the optimal solution to (P_0^ϵ).

Here is a more formal way of explaining this. We shall denote $\mathbf{x}_\mathcal{S}$ and $\mathbf{A}_\mathcal{S}$ the portions of $\mathbf{x}$ and $\mathbf{A}$ that contain the support $\mathcal{S}$ elements/columns, respectively. Suppose that $\mathbf{x}$ is a sparse candidate solution to this problem over the support $\mathcal{S}$, with $\|\mathbf{x}\|_0 = |\mathcal{S}|$, and it satisfies the constraint, $\|\mathbf{b} - \mathbf{A}_\mathcal{S}\mathbf{x}_\mathcal{S}\|_2 \le \epsilon$.

If it so happens that $\mathbf{x}_\mathcal{S}$ is also the minimizer of the term $f_\mathcal{S}(\mathbf{z}) = \|\mathbf{b} - \mathbf{A}_\mathcal{S}\mathbf{z}\|_2$, and $f_\mathcal{S}(\mathbf{x}_\mathcal{S}^{opt}) = \epsilon$, we can propose no alternative solution over this support, since any perturbation around $\mathbf{x}_\mathcal{S}$ leads to an increase in this term and thus violation of the constraint. In terms of Figure 5.1, this case takes place when the closest point to $\mathbf{b}$ on the green line is $\mathbf{A}\mathbf{x}_0$, or put differently, if the distortion $\mathbf{e} = \mathbf{b} - \mathbf{A}_\mathcal{S}\mathbf{x}_\mathcal{S}$ is orthogonal to the columns of $\mathbf{A}_\mathcal{S}$.[1] In all other cases, the fact that $\min_\mathbf{z} f_\mathcal{S}(\mathbf{z}) < \epsilon$ implies an ability to perturb the so-called optimal solution $\mathbf{x}_\mathcal{S}$ in a way that preserves its feasibility and the support, and thus we get a set of solutions that are as good as $\mathbf{x}$.

[1] As the minimizer of $f_\mathcal{S}(\mathbf{z}) = \|\mathbf{b} - \mathbf{A}_\mathcal{S}\mathbf{z}\|_2$, the vector $\mathbf{x}_\mathcal{S}$ should satisfy $\mathbf{A}_\mathcal{S}^T(\mathbf{b} - \mathbf{A}_\mathcal{S}\mathbf{x}_\mathcal{S}) = \mathbf{A}_\mathcal{S}^T\mathbf{e} = \mathbf{0}$.

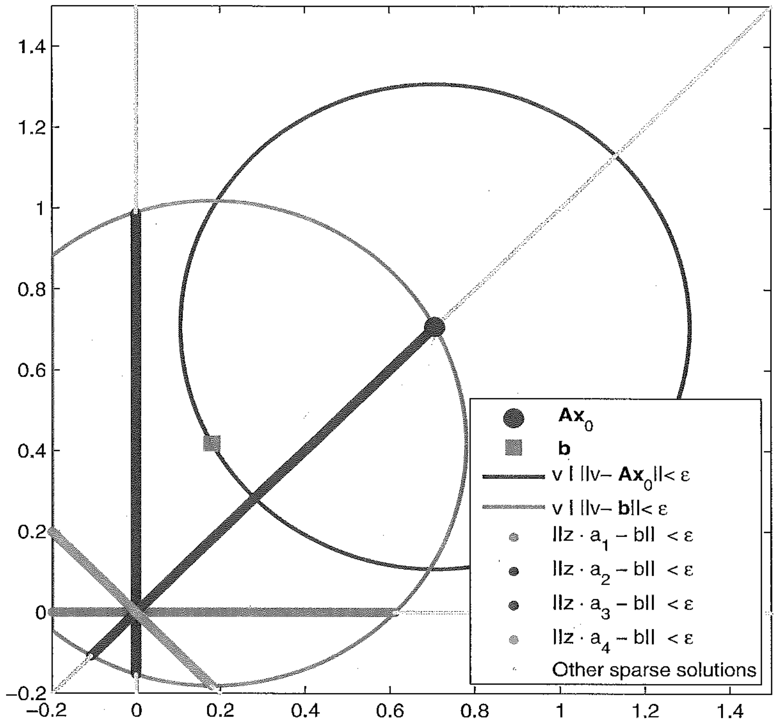


Fig. 5.2 A 2D demonstration of the lack of uniqueness for the noisy case, as shown in Figure 5.1, but with a stronger noise, that permits alternative solutions with a different support.

Furthermore, if some of the non-zero entries in $\mathbf{x}$ are small enough, this perturbation may null them, leading to a sparser solution.

5.2.2 Theoretical Study of the Stability of (P_0^ϵ)

So, returning to the original question we have posed, instead of claiming uniqueness of a sparse solution, we replace this with a notion of stability – a claim that if a sufficiently sparse solution is found, then all alternative solutions necessarily resides very close to it. The following analysis, taken from the work by Donoho, Elad, and Temlyakov, leads to a stability claim of this sort.

We start by returning to the definition of the *spark* and extending it by considering a relaxed notion of linear-dependency. In the noiseless case we considered two competing solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ to the linear system $\mathbf{Ax} = \mathbf{b}$, and this led to the relation $\mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{Ad} = \mathbf{0}$. This motivates a study of the sparsity of vectors $\mathbf{d}$ in the null-space of $\mathbf{A}$, which naturally leads to the definition of the *spark*.

Following the same rationale, we should consider now two feasible solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ to the requirement $\|\mathbf{Ax} - \mathbf{b}\|_2 \le \epsilon$. Considering $\mathbf{b}$ as the center of a sphere of radius ϵ, both $\mathbf{Ax}_1$ and $\mathbf{Ax}_2$ reside in it or on its surface. Thus, the distance between

$\text{Spark}_\eta(A)$

The smallest s such that

$$\min_{\Omega} \sigma_s(A_\Omega) \le \eta$$

$A_\Omega \in \mathbb{R}^{m \times s}$ is a submatrix of A.

The s^{th} singular value

search through all possible submatrices

Example

$$A = \begin{array}{c} \begin{pmatrix} | & | & | & | & | \\ | & | & | & | & | \\ | & | & | & | & | \end{pmatrix} \\ \begin{array}{ccccc} 1 & 0.02 & 0.01 & 0.01 & 0 \end{array} \end{array}$$

singular value

When $\eta = 0$, then $\text{spark}_\eta(A)$ = smallest # columns that are linearly dependent. (which is 5 in this ex.)

When $\eta = 0.01$, then $\text{spark}_\eta(A)$ = smallest # of columns that are needed to guarantee the s^{th} singular is ~~th~~ less than 0.01. (which is 3 in this ex.)

This is not completely correct. From the definition of $\text{spark}_\eta(A)$, we also need to shuffle the submatrices to find out the combination that minimizes the s^{th} singular value.

Remark :

$\text{Spark}_\eta(A)$ is monotonically decreasing in η.

That is, if $\eta \ge \eta'$, then $\text{spark}_\eta(A) \le \text{spark}_{\eta'}(A)$

(You set the threshold higher, of course you will ~~have~~ be easier to find dependent columns)

Theorem 5.1

Let $D \geq 0$, $\epsilon \geq 0$. Let $\eta = \frac{2\epsilon}{D}$. Suppose

$\|x_1\|_0 \leq \frac{1}{2}\operatorname{spark}_\eta(A)$, $\|Ax_1 - b\|_2 \leq \epsilon$,

$\|x_2\|_0 \leq \frac{1}{2}\operatorname{spark}_\eta(A)$, $\|Ax_2 - b\|_2 \leq \epsilon$,

then $\|x_1 - x_2\|_2 \leq D$.

Interpretation :

$\|x_i\|_0 \leq \frac{1}{2}\operatorname{spark}_\eta(A)$ $(i=1,2)$: # x_1 and x_2 are both sufficiently sparse

$\|Ax_i - b\|_2 \leq \epsilon$: x_1 and x_2 are both feasible.

Then $\|x_1 - x_2\|_2 \leq D$: They have to be close.

Or in other words, if they are sparse, then they should be close.

Theorem 5.2

Let $\|x_0\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(A)}\right)$ and $\|Ax_0 - b\| \leq \epsilon$.

Let $x_0^\epsilon = \operatorname{argmin}_x \|x\|_0$ s.t. $\|Ax - b\| \leq \epsilon$.

Then, $$\|x_0^\epsilon - x_0\|_2^2 \leq \frac{4\epsilon^2}{1 - \mu(A)(2\|x_0\|_0 - 1)}.$$

- $\|x_0^\epsilon - x_0\|_2^2 = 0$ when $\epsilon = 0$.
- if x_0 is sparse, then $2\|x_0\|_0 - 1$ is small. And so $1 - \mu(A)(2\|x_0\|_0 - 1)$ is close to 1. But if x is dense, then $\|x_0^\epsilon - x_0\|_2^2 \leq$ a larger upper bound. So x_0^ϵ could be further away from x_0.