ECE 645: Estimation Theory
Spring 2015
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Mid Term Solution

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Name: ____________________ PUID: __________

By signing your name below, you certify that you have neither given nor received unauthorized aid on this exam.

Signature: ______________________________

Problem 1. (50 POINTS)
Consider two hypotheses

$$\begin{aligned} H_0 &: \; Y \sim \mathcal{N}(\mu_0, \sigma^2), \\ H_1 &: \; Y \sim \mathcal{N}(\mu_1, \sigma^2), \end{aligned} \tag{1}$$

where σ^2 is known and fixed. Assume $\mu_0 < \mu_1$.

(a) Assume uniform cost and prior (π_0, π_1), determine the Bayes' decision rule if we observe $Y = y$. Express your answer in terms of π_0, π_1, μ_0, μ_1 and σ.

(b) Derive the Neyman-Pearson rule for a significance level α. Express your answer in terms of the $\Phi(\cdot)$ function, α, μ_0 and σ.

Solution 1.

(a) The likelihood functions are

$$f_0(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\mu_0)^2}{2\sigma^2}\right\}$$
$$f_1(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\mu_1)^2}{2\sigma^2}\right\},$$

Thus, the log-likelihood ratio is

$$\ell(y) = \ln \frac{f_1(y)}{f_0(y)} = \frac{-1}{2\sigma^2}\left[-2(\mu_1 - \mu_0)y + \mu_1^2 - \mu_0^2\right].$$

Let τ be a threshold. Then, $\ell(y) > \tau$ implies that

$$2(\mu_1 - \mu_0)y - (\mu_1^2 - \mu_0^2) > 2\sigma^2\tau,$$

and hence

$$y > \frac{2\sigma^2\tau + (\mu_1^2 - \mu_0^2)}{2(\mu_1 - \mu_0)}.$$

For uniform cost Bayesian setting, $\tau = \ln \frac{\pi_0}{\pi_1}$. Therefore, we have

$$y \lessgtr_{H_1}^{H_0} \frac{2\sigma^2 \ln \frac{\pi_0}{\pi_1} + (\mu_1^2 - \mu_0^2)}{2(\mu_1 - \mu_0)}.$$

(b) Neyman-Pearson takes the form

$$\delta(y) = \begin{cases} 1, & \text{if } \ell(y) > \tau, \\ \gamma, & \text{if } \ell(y) = \tau, \\ 0, & \text{if } \ell(y) < \tau. \end{cases}$$

By letting

$$\eta = \frac{2\sigma^2\tau + (\mu_1^2 - \mu_0^2)}{2(\mu_1 - \mu_0)},$$

we can show that

$$\begin{aligned} \Psi(\tau) &\stackrel{\text{def}}{=} \int_{\ell(y)>\tau} f_0(y)dy \\ &= \int_{y>\eta} f_0(y)dy \\ &= \int_{\eta}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\mu_0)^2}{2\sigma^2}\right\} dy \\ &= 1 - \int_{-\infty}^{\eta} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\mu_0)^2}{2\sigma^2}\right\} dy \\ &= 1 - \Phi\left(\frac{\eta - \mu_0}{\sigma}\right). \end{aligned}$$

Therefore, $\alpha = \Psi(\tau)$ implies that

$$\eta = \sigma\Phi^{-1}(1-\alpha) + \mu_0.$$

Since $\Psi(\eta)$ is a continuous function, $\gamma = 0$. Therefore, the Neyman Peason test is

$$\delta(y) = \begin{cases} 1, & \text{if } y > \sigma\Phi^{-1}(1-\alpha) + \mu_0, \\ 0, & \text{if } y \le \sigma\Phi^{-1}(1-\alpha) + \mu_0. \end{cases}$$

Problem 2. (50 POINTS)
Let $\boldsymbol{Y} = [Y_1, \ldots, Y_n]^T$ be a random vector such that $\boldsymbol{Y} = \theta\boldsymbol{s} + \boldsymbol{V}$, where $\boldsymbol{V} \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$, $\boldsymbol{s} = [s_1, \ldots, s_n]^T$ is a known vector, and θ is a deterministic unknown parameter. Assume $\theta \in \mathbb{R}$.

(a) Show that the conditional distribution $f_\theta(\boldsymbol{y})$ belongs to the exponential family. Hence determine the complete sufficient statistic, and derive the MVUE.

(b) Determine the MLE, i.e., $\widehat{\theta}_{ML}(\boldsymbol{Y})$, and the Cramer Rao Lower Bound. By evaluating $\text{Var}\left(\widehat{\theta}_{ML}(\boldsymbol{Y})\right)$, show that $\widehat{\theta}_{ML}(\boldsymbol{Y})$ achieves the equality of the CRLB.

Solution 2.

(a)

$$\begin{aligned} f_\theta(\boldsymbol{y}) &= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{n/2} \exp\left\{\frac{-1}{2\sigma^2}\|\boldsymbol{y}-\theta\boldsymbol{s}\|^2\right\} \\ &= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{n/2} \exp\left\{-\frac{\theta^2\|\boldsymbol{s}\|^2}{2\sigma^2}\right\} \exp\left\{\frac{\theta}{\sigma^2}\cdot\boldsymbol{y}^T\boldsymbol{s}\right\} \exp\left\{-\frac{1}{2\sigma^2}\|\boldsymbol{y}\|^2\right\}. \end{aligned}$$

By identifying

$$\begin{aligned} C(\theta) &= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{n/2} \exp\left\{-\frac{\theta^2\|\boldsymbol{s}\|^2}{2\sigma^2}\right\} \\ Q(\theta) &= \frac{\theta}{\sigma^2} \\ T(\boldsymbol{y}) &= \boldsymbol{y}^T\boldsymbol{s} \\ h(\boldsymbol{y}) &= \exp\left\{-\frac{1}{2\sigma^2}\|\boldsymbol{y}\|^2\right\}, \end{aligned}$$

we see that $f_\theta(\boldsymbol{y})$ is in the exponential family. Since $\theta \in \mathbb{R}$ and $\mathbb{R}$ contains a rectangle, $T(\boldsymbol{y})$ is a complete sufficient statistic. Now, consider $\mathbb{E}[T(\boldsymbol{Y})]$. We can show that

$$\mathbb{E}[T(\boldsymbol{Y})] = \mathbb{E}[\boldsymbol{Y}^T\boldsymbol{s}] = \theta\|\boldsymbol{s}\|^2.$$

Thus, if we let

$$\widehat{\theta}_{MVUE}(\boldsymbol{Y}) = \frac{T(\boldsymbol{Y})}{\|\boldsymbol{s}\|^2} = \frac{\boldsymbol{Y}^T\boldsymbol{s}}{\|\boldsymbol{s}\|^2},$$

then we have $\mathbb{E}[\widehat{\theta}_{MVUE}(\boldsymbol{Y})] = \theta$. So $\widehat{\theta}_{MVUE}(\boldsymbol{Y})$ is unbiased. Since $\widehat{\theta}_{MVUE}(\boldsymbol{Y})$ is a function of the complete sufficient statistic $T(\boldsymbol{Y})$, it must be MVUE.

(b) Take log on the likelihood function yields

$$\ln f_\theta(\boldsymbol{y}) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\|\boldsymbol{y}-\theta\boldsymbol{s}\|^2.$$

Take first order derivative yields

$$0 = \frac{\partial}{\partial\theta}\ln f_\theta(\boldsymbol{y}) = \frac{1}{\sigma^2}\boldsymbol{s}^T(\boldsymbol{y}-\theta\boldsymbol{s}).$$

Thus,

$$\widehat{\theta}_{ML}(\boldsymbol{y}) = \frac{\boldsymbol{s}^T\boldsymbol{y}}{\|\boldsymbol{s}\|^2}.$$

The Fisher Information is

$$\begin{aligned} I(\theta) &= \mathbb{E}\left[-\frac{\partial^2}{\partial\theta^2}\ln f_\theta(\boldsymbol{y})\right] \\ &= \frac{\|\boldsymbol{s}\|^2}{\sigma^2}. \end{aligned}$$

Since $\widehat{\theta}_{ML}(\boldsymbol{Y})$ is unbiased, the CRLB is

$$\mathrm{Var}(\widehat{\theta}_{ML}(\boldsymbol{Y})) \geq \frac{\sigma^2}{\|\boldsymbol{s}\|^2}.$$

To show that $\widehat{\theta}_{ML}(\boldsymbol{Y})$ achieves the lower bound of CRLB, we note that

$$\begin{aligned}
\operatorname{Var}(\widehat{\theta}_{ML}(\boldsymbol{Y})) &= \operatorname{Var}\left(\frac{\boldsymbol{Y}^T\boldsymbol{s}}{\|\boldsymbol{s}\|^2}\right) \\
&= \frac{1}{\|\boldsymbol{s}\|^4}\operatorname{Var}\left(\sum_{k=1}^{n} s_k Y_k\right) \\
&= \frac{1}{\|\boldsymbol{s}\|^4}\sum_{k=1}^{n} s_k^2 \operatorname{Var}(Y_k) \\
&= \frac{1}{\|\boldsymbol{s}\|^4}\sum_{k=1}^{n} s_k^2 \sigma^2 = \frac{\sigma^2}{\|\boldsymbol{s}\|^2}.
\end{aligned}$$

Problem 3. (Bouns, 10 points)
Consider a Poisson distribution with parameter $\lambda > 0$ with

$$f_Y(y) = \frac{\lambda^y}{y!}e^{-\lambda}. \tag{2}$$

It is given that the cumulant generating function is $\mu_Y(s) \stackrel{\text{def}}{=} \log \mathbb{E}[e^{sY}] = \lambda(e^s - 1)$. Let $Y_1, \ldots, Y_n$ be a sequence of observations. Derive the large-deviation bound for

$$\mathbb{P}\left[\sum_{k=1}^{n} Y_k \geq n\lambda e\right], \tag{3}$$

where $e \approx 2.718$ is the natural number.

Solution 3.
Let $g(s) = st - \mu_Y(s)$, we have

$$\frac{\partial}{\partial s} g(s) = t - \lambda e^s = 0.$$

Thus, $s^* = \ln(t/\lambda)$ is the optimal point. Hence,

$$\begin{aligned}
\varphi(t) = g(s^*) &= \ln(t/\lambda)t - \lambda\left(e^{\ln(t/\lambda)} - 1\right) \\
&= \ln(t/\lambda)t - (t - \lambda).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{P}\left(\sum_k Y_k \geq n\lambda e\right) &\leq e^{-sn\lambda e}\left(\mathbb{E}\left[e^{sY_1}\right]\right)^n \\
&= e^{-n(s\lambda e - \mu_Y(s))}.
\end{aligned}$$

Since $\varphi(\lambda e) = \ln(\lambda e/\lambda)\lambda e - (\lambda e - \lambda) = \lambda$, we have

$$\mathbb{P}\left(\sum_k Y_k \geq n\lambda e\right) \leq e^{-n\lambda}.$$