ECE 645: Estimation Theory
Spring 2015
Instructor: Prof. Stanley H. Chan

PURDUE UNIVERSITY

Mid Term

Mar 13, 2015

Name: _____ PUID: _____

By signing your name below, you certify that you have neither given nor received unauthorized aid on this exam.

Signature: _____

Problem 1. (50 POINTS)
Consider two hypotheses

$$\begin{aligned} H_0: &\quad Y \sim \mathcal{N}(\mu_0, \sigma^2), \\ H_1: &\quad Y \sim \mathcal{N}(\mu_1, \sigma^2), \end{aligned} \tag{1}$$

where σ^2 is known and fixed.

(a) Assume uniform cost and prior (π_0, π_1), determine the Bayes' decision rule if we observe $Y = y$. Express your answer in terms of π_0, π_1, μ_0, μ_1 and σ.

(b) Derive the Neyman-Pearson rule for a significance level α. Express your answer in terms of the $\Phi(\cdot)$ function, α, μ_0 and σ.

Problem 2. (50 POINTS)
Let $\boldsymbol{Y} = [Y_1, \ldots, Y_n]^T$ be a random vector such that $\boldsymbol{Y} = \theta \boldsymbol{s} + \boldsymbol{V}$, where $\boldsymbol{V} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$, $\boldsymbol{s} = [s_1, \ldots, s_n]^T$ is a known vector, and θ is a deterministic unknown parameter. Assume $\theta \in \mathbb{R}$.

(a) Show that the conditional distribution $f_\theta(\boldsymbol{y})$ belongs to the exponential family. Hence determine the complete sufficient statistic, and derive the MVUE.

(b) Determine the MLE, i.e., $\widehat{\theta}_{ML}(\boldsymbol{Y})$, and the Cramer Rao Lower Bound. By evaluating $\text{Var}\left(\widehat{\theta}_{ML}(\boldsymbol{Y})\right)$, show that $\widehat{\theta}_{ML}(\boldsymbol{Y})$ achieves the equality of the CRLB.

Problem 3. (BONUS, 10 POINTS)
Consider a Poisson distribution with parameter $\lambda > 0$ with

$$f_Y(y) = \frac{\lambda^y}{y!} e^{-\lambda}. \tag{2}$$

It is given that the cumulant generating function is $\mu_Y(s) \stackrel{\text{def}}{=} \log \mathbb{E}[e^{sY}] = \lambda(e^s - 1)$. Let $Y_1, \ldots, Y_n$ be a sequence of observations. Derive the large-deviation bound for

$$\mathbb{P}\left[\sum_{k=1}^{n} Y_k \geq n\lambda e\right], \tag{3}$$

where $e \approx 2.718$ is the natural number.

Problem 4. (BONUS, 5 POINTS)
We still need a few more volunteers to help us typeset the note. If you volunteer, you will automatically gain 5 bonus points to this mid term. Just write: "Yes", or "No".