Conditions for CRLB Equality

Proposition (Poor Example IV.C.4)

An estimator $\hat{\theta}(y)$ achieves CRLB equality if and only if $\hat{\theta}(y)$ is a sufficient statistic of a one parameter exponential family.

Pf Suppose CRLB equality holds, then by Cauchy-Schwarz, we have

$$\frac{\partial}{\partial\theta} \ln f_\theta(y) = k(\theta)\left[\hat{\theta}(y) - \mathbb{E}[\hat{\theta}(Y)]\right].$$

$$\Rightarrow \quad f_\theta(y) = \exp\left\{\int_a^\theta k(\theta')\left(\hat{\theta}(y) - \mathbb{E}[\hat{\theta}(Y)]\right)d\theta'\right\} h(y)$$

$$= \underbrace{\exp\left\{-\int_a^\theta k(\theta')\mathbb{E}[\hat{\theta}(Y)]\,d\theta'\right\}}_{C(\theta)} \exp\left\{\underbrace{\hat{\theta}(y)}_{T(y)}\underbrace{\int_a^\theta k(\theta')\,d\theta'}_{Q(\theta)}\right\} h(y)$$

Therefore,

$$\hat{\theta}(y) = T(y).$$

Since $T(y)$ is a suff stat for exponential family, we are done.

Suppose $\hat{\theta}(y)$ is a suff stat of a one parameter exponential family. Then

$$f_\theta(y) = C(\theta)\exp\left\{Q(\theta)T(y)\right\}h(y), \text{ where } \hat{\theta}(y) = T(y).$$

where $C(\theta)$ is a normalization constant s.t.

$$C(\theta) = \left(\int \exp\left\{Q(\theta)T(y)\right\}h(y)\,dy\right)^{-1}$$

Therefore,

$$\ln f_\theta(y) = Q(\theta)T(y) + \ln h(y) - \ln \int \exp(Q(\theta)T(y))h(y)\,dy$$

$$\frac{\partial}{\partial\theta}\ln f_\theta(y) = Q'(\theta)T(y) - \frac{\int T(y)\,e^{Q(\theta)T(y)}h(y)\,dy}{\int e^{Q(\theta)T(y)}h(y)\,dy}$$

$$= Q'(\theta)\Big[T(y) - \underbrace{\mathbb{E}[T(Y)]}_{=\int T(y)f_\theta(y)\,dy}\Big]$$

So $I(\theta) = \mathbb{E}\Big[\Big(\frac{\partial}{\partial\theta}\ln f_\theta(Y)\Big)^2\Big]$

$$= \mathbb{E}\Big[\Big(Q'(\theta)\big[T(Y) - \mathbb{E}[T(Y)]\big]\Big)^2\Big]$$

$$= (Q'(\theta))^2\,\mathrm{Var}(T(Y)).$$

hence CRLB is

$$\mathrm{Var}(\hat\theta(Y)) \geq \frac{\Big(\frac{\partial}{\partial\theta}\mathbb{E}[\hat\theta(Y)]\Big)^2}{I(\theta)}.$$

Suppose $\hat\theta(Y) = T(Y)$, then

$$\mathbb{E}[\hat\theta(Y)] = \frac{\int T(y)\,e^{Q(\theta)T(y)}h(y)\,dy}{\int e^{Q(\theta)T(y)}h(y)\,dy}$$

$$\Rightarrow \frac{\partial}{\partial\theta}\mathbb{E}[\hat\theta(Y)] = Q'(\theta)\Bigg[\frac{\int e^{Q(\theta)T(y)}h(y)\,dy\int T(y)^2 e^{Q(\theta)T(y)}h(y)\,dy}{\Big(\int e^{Q(\theta)T(y)}h(y)\,dy\Big)^2} - \Big(\int T(y)\,e^{Q(\theta)T(y)}h(y)\,dy\Big)^2\Bigg]$$

$$= Q'(\theta)\,\mathrm{Var}(T(Y)).$$

So $\dfrac{\Big(\frac{\partial}{\partial\theta}\mathbb{E}[\hat\theta(Y)]\Big)^2}{I(\theta)} = \dfrac{(Q'(\theta))^2(\mathrm{Var}(T(Y)))^2}{(Q'(\theta))^2\,\mathrm{Var}(T(Y))} = \mathrm{Var}(T(Y))$.

Hence, CRLB equality is achieved.
This completes the proof.

Discussion

① if $T(y)$ is a complete suff stat of an one-parameter exponential family, ~~then~~ and if $\hat{\theta}(y) = T(y)$, then $\hat{\theta}(y)$ is MVUE and it achieves CRLB equality.

② Consider the ~~expon~~ example

$$Y_k \overset{iid}{\sim} \exp(\theta), \quad f_\theta(\underline{y}) = \underbrace{\theta^n}_{C(\theta)} e^{\underbrace{-\theta T(\underline{y})}_{Q(\theta)T(\underline{y})}},$$

where $T(\underline{y}) = \sum y_k$.

We have shown that

$$\hat{\theta}_{ML}(\underline{Y}) = \frac{n}{\sum Y_k} = \frac{n}{T(\underline{Y})}.$$

However, $f_\theta(\underline{y})$ cannot be written in terms of $\hat{\theta}_{ML}(\underline{y})$ and maintain the exponential family structure.
So $\hat{\theta}_{ML}(\underline{y})$ cannot achieve CRLB equality.

How about $\hat{\theta}_{MVUE}(\underline{Y}) = \dfrac{n-1}{T(\underline{Y})}$?

Same as above, $f_\theta(\underline{y})$ cannot be written in terms of $\hat{\theta}_{MVUE}(\underline{y})$. So CRLB equality cannot be achieved.