Conditions for CRLB Equality

Proposition (Poor Example IV.C.4)
An estimator \( \hat{\theta}(y) \) achieves CRLB equality if and only if \( \hat{\theta}(y) \) is a sufficient statistic of a one parameter exponential family.

Proof: Suppose CRLB equality holds, then by Cauchy-Schwarz, we have

\[
\frac{a}{\theta} \ln f_\theta(y) = k(\theta) \left[ \hat{\theta}(y) - \mathbb{E}[\hat{\theta}(y)] \right].
\]

\[\Rightarrow f_\theta(y) = \exp \left\{ \mathbb{E}[\hat{\theta}(y)] - \mathbb{E}[\hat{\theta}(y)] \right\} \exp \left\{ \hat{\theta}(y) \mathbb{E}[\hat{\theta}(y)] - \mathbb{E}[\hat{\theta}(y)] \right\} \exp \left\{ \hat{\theta}(y) \mathbb{E}[\hat{\theta}(y)] - \mathbb{E}[\hat{\theta}(y)] \right\}
\]

\[= \exp \left\{ -\int_a^\theta k(\theta') \mathbb{E}[\hat{\theta}(y)] \, d\theta' \right\} \exp \left\{ \hat{\theta}(y) \int_a^\theta k(\theta') \, d\theta' \right\} h(y)
\]

Therefore,

\[\hat{\theta}(y) = T(y).
\]

Since \( T(y) \) is a suff stat for exponential family, we are done.

Suppose \( \hat{\theta}(y) \) is a suff stat of a one parameter exponential family. Then

\[f_\theta(y) = C(\theta) \exp \left\{ Q(\theta) T(y) \right\} h(y), \text{ where } \hat{\theta}(y) = T(y),
\]

where \( C(\theta) \) is a normalization constant s.t.

\[C(\theta) = \left( \int \exp \left\{ Q(\theta) T(y) \right\} h(y) \, dy \right)^{-1}
\]
Therefore,
\[
\ln f_0(y) = Q(\theta) T(y) + \ln h(y) - \ln \int \exp(Q(\theta) T(y)) h(y) \, dy
\]
\[
\frac{\partial}{\partial \theta} \ln f_0(y) = Q'(\theta) T(y) - \frac{\int T(y) e^{Q(\theta) T(y)} h(y) \, dy}{\int e^{Q(\theta) T(y)} h(y) \, dy}
\]
\[
= Q'(\theta) \left[ T(y) - \mathbb{E}[T(Y)] \right]
\]
\[
\uparrow
\]
\[
I(\theta) = \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \ln f_0(y) \right)^2 \right]
\]
\[
= \mathbb{E} \left[ Q'(\theta) \left( T(y) - \mathbb{E}[T(Y)] \right)^2 \right]
\]
\[
= (Q'(\theta))^2 \text{Var}(T(Y)),
\]

hence CRLB is
\[
\text{Var}(\hat{\theta}(Y)) \geq \frac{\frac{\partial}{\partial \theta} \mathbb{E}[\hat{\theta}(Y)]^2}{I(\theta)}.
\]

Suppose \( \hat{\theta}(Y) = T(Y) \), then
\[
\mathbb{E}[\hat{\theta}(Y)] = \frac{\int T(y) e^{Q(\theta) T(y)} h(y) \, dy}{\int e^{Q(\theta) T(y)} h(y) \, dy}
\]
\[
\Rightarrow \frac{\partial}{\partial \theta} \mathbb{E}[\hat{\theta}(Y)] = Q'(\theta) \left[ \frac{\int e^{Q(\theta) T(y)} h(y) \, dy}{\int e^{Q(\theta) T(y)} h(y) \, dy} \right]
\]
\[
\uparrow
\]
\[
= (Q'(\theta))^2 \left( \text{Var}(T(Y)) \right)
\]
\[
\Rightarrow \frac{\partial}{\partial \theta} \mathbb{E}[\hat{\theta}(Y)]^2
\]
\[
= \left( \frac{\partial}{\partial \theta} \mathbb{E}[\hat{\theta}(Y)] \right)^2 = \frac{(Q'(\theta))^2 (\text{Var}(T(Y)))^2}{I(\theta)} = \frac{Q'(\theta)}{\text{Var}(T(Y))}.
\]
Hence, CRLB equality is achieved. This completes the proof.

Discussion

1. If $T(y)$ is a complete suff stat of an one-parameta exponential family, then if $\hat{\theta}(y) = T(y)$, then $\hat{\theta}(y)$ is MVUE and it achieves CRLB equality.

2. Consider the exponential example

$$Y_k \overset{iid}{\sim} \exp(\theta), \quad f_\theta(y) = \frac{\theta^n}{\Gamma(n)} e^{-\theta T(y)}$$

where $T(y) = \sum y_k$.

We have shown that

$$\hat{\theta}_{ML}(Y) = \frac{n}{\sum y_k} = \frac{n}{T(Y)}.$$

However, $f_\theta(y)$ cannot be written in terms of $\hat{\theta}_{ML}(y)$ and maintain the exponential family structure. So $\hat{\theta}_{ML}(y)$ cannot achieve CRLB equality.

How about $\hat{\theta}_{MVUE}(Y) = \frac{n-i}{T(Y)}$?

Same as above, $f_\theta(y)$ cannot be written in terms of $\hat{\theta}_{MVUE}(y)$. So CRLB equality cannot be achieved.