Lecture 21

Effiziency of MLE

1. Efficient Estimator

Def An estimator $\hat{\theta}(\underline{y})$ is efficient if it achieves equality in CRLB.

Example

$$Y_k \overset{iid}{\sim} N(\theta, \sigma^2), \quad k=1,\dots,n$$

$$\hat{\theta}(\underline{Y}) = \frac{1}{n}\sum_{k=1}^{n} Y_k .$$

We can show that CRLB is

$$\mathrm{Var}(\hat{\theta}(\underline{Y})) \geq \frac{\sigma^2}{n}$$

Question: Is $\hat{\theta}(\underline{Y})$ efficient?

Solution:

$$\mathrm{Var}(\hat{\theta}(\underline{Y})) = \mathrm{Var}\left(\frac{1}{n}\sum_{k=1}^{n} Y_k\right)$$

$$= \frac{\sigma^2}{n}.$$

So CRLB equality is achieved, thus $\hat{\theta}(\underline{Y})$ is efficient.

2. Efficient Estimator and MVUE

Proposition

If an estimator $\hat{\theta}(y)$ is unbiased and is efficient, then it must be MVUE.

Pf: By CRLB,

$$\mathrm{Var}(\hat{\theta}(y)) \leq \mathrm{Var}(\tilde{\theta}(y))$$

for any $\tilde{\theta}(y)$. Therefore $\hat{\theta}(y)$ must be MV. Since $\hat{\theta}(y)$ is also unbiased, it is MVUE.

Remark: The converse is not true in general. That is, MVUE does not need to ~~ach~~ be efficient.

Example

Y_k iid exponential $\left(\frac{1}{\theta}\right)$.

Find $\hat{\theta}_{ML}$, $\hat{\theta}_{MVUE}$, and CRLB.

Solution:

$$f_\theta(y_k) = \theta e^{-\theta y_k}$$

So $$f_\theta(\underline{y}) = \prod_{k=1}^{n} f_\theta(y_k) = \underbrace{\theta^n e^{-\theta \sum_{k=1}^{n} y_k}}_{\text{exponential family}}$$

So $\sum y_k$ is suff stat.

$\hat{\theta}_{ML}$:

$$\frac{\partial}{\partial\theta}\ln f_\theta(\underline{y}) = \frac{\partial}{\partial\theta}\left[n\ln\theta - \theta\sum y_k\right]$$

$$= \frac{n}{\theta} - \sum y_k = 0$$

$$\Rightarrow \quad \hat{\theta}_{ML} = \frac{n}{\sum_{k=1}^{n} y_k}.$$

$\mathbb{E}[\hat{\theta}_{ML}] = ?$ Let $T(\underline{Y}) = \sum_{k=1}^{n} Y_k$.

By moment generating function we can show that

$$f_T(t) = \frac{\theta^n t^{n-1}}{(n-1)!} e^{-\theta t}$$

So, $\mathbb{E}[\hat{\theta}_{ML}] = \int_0^\infty \frac{n}{t}\frac{\theta^n t^{n-1}}{(n-1)!} e^{-\theta t}\, dt$

$$= \underbrace{\int_0^\infty \frac{\theta^{n-1} t^{n-2}}{(n-2)!} e^{-\theta t}\, dt}_{\text{mean of poisson}} \cdot \frac{n}{(n-1)!}\,\theta\,(n-2)!$$

$$= \frac{n}{n-1}\theta. \quad \text{So } \hat{\theta}_{ML} \text{ is biased.}$$

$$Var(\hat{\theta}_{ML}) = \frac{\theta^2 n^2}{(n-1)^2(n-2)}.$$

CRLB:

$$I(\theta) = -\frac{\partial^2}{\partial\theta^2}\ln f_\theta(\underline{y}) = \frac{n}{\theta^2}$$

So $Var(\hat{\theta}(\underline{Y})) \geq \frac{\left(\frac{\partial}{\partial\theta}\mathbb{E}[\hat{\theta}(\underline{Y})]\right)^2}{I(\theta)}$.

Thus, $\left(\frac{\partial}{\partial\theta}\mathbb{E}[\hat{\theta}_{ML}(\underline{Y})]\right)^2 = \left(\frac{\partial}{\partial\theta}\left(\frac{n}{n-1}\theta\right)\right)^2 = \frac{n^2}{(n-1)^2}$.

$$\Rightarrow Var(\hat{\theta}_{ML}(\underline{Y})) \geq \frac{\frac{n^2}{(n-1)^2}}{\frac{n}{\theta^2}} = \frac{n}{(n-1)^2}\theta^2$$

Since $Var(\hat{\theta}_{ML}(\underline{Y})) = \frac{n}{(n-1)^2}\theta^2 \cdot \frac{n}{(n-2)}$, $\hat{\theta}_{ML}(\underline{Y})$ is NOT efficient.

$\hat{\theta}_{MVUE}$: Note that $T(\underline{Y}) = \sum Y_k$ is a complete suff stat.

Also, $\hat{\theta}_{ML} = \frac{n}{T(\underline{Y})}$.

But $\mathbb{E}[\hat{\theta}_{ML}] = \frac{n}{n-1}\theta$. So

Let $\tilde{\theta}(\underline{Y}) = \frac{n-1}{T(\underline{Y})}$.

Then $\mathbb{E}[\tilde{\theta}(\underline{Y})] = \mathbb{E}\left[\frac{n}{T(\underline{Y})} \cdot \frac{n-1}{n}\right]$

$= \mathbb{E}\left[\hat{\theta}_{ML} \cdot \frac{n-1}{n}\right]$

$= \theta$.

Since $\tilde{\theta}(\underline{Y}) = \frac{n-1}{T(\underline{Y})}$ is a function of complete suff stat., $\tilde{\theta}$ is MVUE.

So $\hat{\theta}_{MVUE} = \frac{n-1}{T(\underline{Y})}$.

$\mathbb{E}[\hat{\theta}_{MVUE}] = \theta$, $Var(\hat{\theta}_{MVUE})$

$= Var\left(\frac{n-1}{n} \cdot \hat{\theta}_{ML}\right)$

$= \left(\frac{n-1}{n}\right)^2 \left(\frac{n^2\theta^2}{(n-1)^2(n-2)}\right)$

$= \frac{\theta^2}{(n-2)}$.

CRLB: $Var(\hat{\theta}_{MVUE}) \geq \frac{1}{I(\theta)} = \frac{\theta^2}{n}$.

So $\hat{\theta}_{MVUE}$ is also not efficient.