Lecture 20

Maximum Likelihood Estimation (III)

Cramer-Rao Lower Bound

1. CRLB Theorem

Under Assumption (1)-(2),

\[
\text{Var} \left( \hat{\theta} (Y) \right) \geq \left( \frac{3 \sigma}{2} \frac{E\left[ \hat{\theta}(Y) \right]}{I(\theta)} \right)^2
\]

for any unbiased estimator \( \hat{\theta} \).

**Proof:**

\[
\text{Var} \left( \hat{\theta} (Y) \right) I(\theta)
= \int \left( \hat{\theta}(Y) - E \left[ \hat{\theta}(Y) \right] \right)^2 f_0(y) dy \cdot \int \frac{1}{\sigma} \left( \frac{3 \sigma}{2} \text{ln} f_0(y) \right)^2 f_0(y) dy
\]

\[
= E\left[ A^2 \right] E\left[ B^2 \right]
\geq E\left[ A B \right]^2 \quad \text{(Cauchy inequality)}
\]

\[
= \left[ \int \left( \hat{\theta}(Y) - E \left[ \hat{\theta}(Y) \right] \right) \frac{3 \sigma}{2} f_0(y) dy \right]^2
\]

\[
= \left[ \int \hat{\theta}(Y) \frac{3 \sigma}{2} f_0(y) dy - E \left[ \hat{\theta}(Y) \right] \int \frac{3 \sigma}{2} f_0(y) dy \right]^2
\]

\[
= \left( \frac{3 \sigma}{2} E \left[ \hat{\theta}(Y) \right] - 0 \right)^2
\]

\[
= \left( \frac{3 \sigma}{2} E \left[ \hat{\theta}(Y) \right] \right)^2.
\]
(2) If \( \hat{\theta} \) is an admissible estimator, then

\[
\frac{3}{3\theta} \ln f_\theta(y) = k(\theta) (\hat{\theta}(y) - E[\hat{\theta}(y)])
\]

\[
\Rightarrow \ln f_\theta(y) = \int_\theta^0 k(\theta') (\hat{\theta}(y) - E[\hat{\theta}(y)]) d\theta' + C(y).
\]

\[
= \int_\theta^0 \frac{k(\theta') E[\hat{\theta}(y)] d\theta'}{\ln A(\theta)} - \frac{\ln h(y)}{Q(\theta)} \hat{\theta}(y) \int_\theta^0 k(\theta') d\theta'.
\]

\[
\Rightarrow f_\theta(y) = A(\theta) \exp \left( Q(\theta) \hat{\theta}(y) \right) h(y).
\]

\[
\Rightarrow f_\theta(y) \text{ is a one-parameter exponential family.}
\]

Conversely, if \( f_\theta(y) = A(\theta) \exp \left( Q(\theta) \hat{\theta}(y) \right) h(y) \),

then CRLE equality is satisfied, when

\( \hat{\theta}(y) \) is a suff. statistic.
Conditions for Equality

\begin{align*}
(1) \quad \frac{\partial}{\partial \theta} \ln f_\theta(y) &= I(\theta)[\hat{\theta}(y) - \theta], \\
\text{if } \hat{\theta}(Y) \text{ is an unbiased estimator.}
\end{align*}

\textbf{Proof:}
\[ \frac{\partial^2}{\partial \theta^2} \ln f_\theta(y) = I'(\theta)(\hat{\theta} - \theta) + I(\theta) \]
\[ \text{So } \mathbb{E} \left[ -\frac{\partial^2}{\partial \theta^2} \ln f_\theta(y) \right] = -I'(\theta)\mathbb{E}[\hat{\theta} - \theta] + I(\theta) = I(\theta). \]

\textbf{Proof:} Cauchy holds when
\[ A = x \cdot B. \quad \mathbb{E}[\hat{\theta}] = 0 \]
\[ \text{So } \frac{\partial}{\partial \theta} \ln f_\theta(y) = k(\theta) \cdot (\hat{\theta}(y) - \theta). \]
\[ \Rightarrow \quad \frac{\partial^2}{\partial \theta^2} \ln f_\theta(y) = k'(\theta)(\hat{\theta}(y) - \theta) - k(\theta) \]
\[ \Rightarrow \quad -\mathbb{E} \left[ \frac{\partial^2}{\partial \theta^2} \ln f_\theta(y) \right] = k(\theta) \]
\[ = I(\theta). \]
\[ \text{So } \frac{\partial}{\partial \theta} \ln f_\theta(y) = I(\theta)(\hat{\theta}(y) - \theta). \]
2. Examples

Example 1

\[ Y_k \text{iid } \mathcal{N}(\theta, \sigma^2), \quad k = 1, \ldots, n \]

Find CRLB for \( \hat{\theta}(Y) = \frac{1}{n} \sum_{i=1}^{n} Y_i \). Is \( \hat{\theta}(Y) \) MVUE?

Solution \( I(\theta) = \frac{n}{\sigma^2} \).

So \( \text{Var}(\hat{\theta}) \geq \left( \frac{\frac{2}{\sigma^2} \mathbb{E}[\hat{\theta}(Y)]}{I(\theta)} \right)^2 \)

If \( \hat{\theta}(Y) = \frac{1}{n} \sum_{i=1}^{n} Y_i \),

then \( \mathbb{E}[\hat{\theta}(Y)] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[Y_i] = \theta \).

So \( \frac{2}{\sigma^2} \mathbb{E}[\hat{\theta}(Y)] = 1 \).

Hence \( \text{Var}(\hat{\theta}) \geq \frac{1}{I(\theta)} = \frac{\sigma^2}{n} \).

\[ \text{Var}(\hat{\theta}) = \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} Y_i \right) \]

\[ = \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}(Y_i) = \frac{\sigma^2}{n} \]

So lower bound is attained at \( \hat{\theta}(Y) = \frac{1}{n} \sum_{i=1}^{n} Y_i \).

Therefore, \( \hat{\theta}(Y) = \frac{1}{n} \sum_{i=1}^{n} Y_i \) is MVUE.
Example 2

\[ Y_k \sim iid \ S_0(k) + N_k \quad k = 1, \ldots, n \]

where \( S_k(\theta) \) is a function of \( k \).

\( N_k \sim N(0, \sigma^2) \).

Find CRLB for any unbiased estimator \( \hat{\theta} \).

Solution:

\[
\ln f_\theta(y) = \frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - S_k(\theta))^2
\]

\[
\frac{\partial}{\partial \theta} \ln f_\theta(y) = -\frac{1}{\sigma^2} \sum_{k=1}^{n} (y_k - S_k(\theta)) \frac{\partial}{\partial \theta} S_k(\theta)
\]

\[
\frac{\partial^2}{\partial \theta^2} \ln f_\theta(y) = \frac{1}{\sigma^2} \sum_{k=1}^{n} \left[ (y_k - S_k(\theta))^2 \frac{\partial^2}{\partial \theta^2} S_k(\theta) \right]
\]

\[
= \frac{1}{\sigma^2} \sum_{k=1}^{n} (y_k - S_k(\theta)) \left[ \frac{\partial^2}{\partial \theta^2} S_k(\theta) \right] - \left[ \frac{\partial}{\partial \theta} S_k(\theta) \right]^2.
\]

\[
\mathbb{E} \left[ \frac{\partial^2}{\partial \theta^2} \ln f_\theta(y) \right] = \frac{1}{\sigma^2} \sum_{k=1}^{n} \left( \frac{\partial}{\partial \theta} S_k(\theta) \right)^2.
\]

\[
\text{So} \quad \text{Var}(\hat{\theta}) \geq \frac{1}{I(\theta)}
\]

\[
= \frac{\sigma^2}{\sum_{k=1}^{n} \left( \frac{\partial}{\partial \theta} S_k(\theta) \right)^2}.
\]

E.g. If \( S_k(\theta) = 0 \), then \( \text{Var}(\hat{\theta}) \geq \frac{\sigma^2}{n} \).

If \( S_k(\theta) = A \cos(2\pi k + \theta) \), then \( \text{Var}(\hat{\theta}) \geq \frac{2\pi^2}{nA^2} \).
3. Extension to Vector

CRLB (Vector)

Under assumption (i)-(2),

\[ \text{Cov}(\hat{\theta}(Y)) = I(\theta)^{-1} \succeq 0, \]

for any unbiased estimator \( \hat{\theta}(Y) \).

\[ I(\theta)_{ij} = -\mathbb{E} \left[ \frac{\partial^2 \ln f_\theta(y)}{\partial \theta_i \partial \theta_j} \right]. \]

Example

\( Y_k = A + Bk + N_k, \quad N_k \sim N(0, \sigma^2) \).

Find CRLB for \((A, B)\), for any unbiased estimator \( \hat{\theta} = (\hat{A}, \hat{B}) \).

Solution:

\[ I(\theta) = \begin{bmatrix}
-\mathbb{E} \left[ \frac{\partial^2 \ln f_\theta(y)}{\partial A^2} \right] & -\mathbb{E} \left[ \frac{\partial^2 \ln f_\theta(y)}{\partial A \partial B} \right] \\
-\mathbb{E} \left[ \frac{\partial^2 \ln f_\theta(y)}{\partial B \partial A} \right] & -\mathbb{E} \left[ \frac{\partial^2 \ln f_\theta(y)}{\partial B^2} \right]
\end{bmatrix} \]

\[ \ln f_\theta(y) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{k=1}^{n} (y_k - A - Bk)^2 \]

\[ \frac{\partial}{\partial A} \ln f_\theta(y) = \frac{1}{\sigma^2} \sum_{k=1}^{n} (y_k - A - Bk) \]

\[ \frac{\partial}{\partial B} \ln f_\theta(y) = \frac{1}{\sigma^2} \sum_{k=1}^{n} k (y_k - A - Bk) \]


Therefore,
\[
\frac{\partial^2}{\partial \theta^2} \ln f_\theta(y) = -\frac{n}{\sigma^2},
\]
\[
\frac{\partial^2}{\partial \theta^2} \ln f_\theta(y) = -\frac{1}{\sigma^2} \sum_{k=1}^{n} k^2
\]
\[
\frac{\partial^2}{\partial \theta \partial \theta} \ln f_\theta(y) = -\frac{1}{\sigma^2} \sum_{k=1}^{n} k
\]

So,
\[
I(\theta) = \frac{1}{\sigma^2} \left[ \sum_{k=1}^{n} k \quad \sum_{k=1}^{n} k^2 \right] = \frac{1}{\sigma^2} \left[ \sum_{k=1}^{n} \frac{n(n-1)}{2} \quad \frac{n(n-1)(2n-1)}{6} \right]
\]

For any unbiased estimator \( \hat{\theta} = (\hat{A}, \hat{B}) \),
\[
\text{Cov}(\hat{\theta}(Y)) \geq I(\theta)^{-1}
\]
\[
= \sigma^2 \begin{bmatrix}
\frac{2}{n(n+1)} & -\frac{6}{n(n+1)} \\
-\frac{6}{n(n+1)} & \frac{12}{n(n^2-1)}
\end{bmatrix}
\]

Therefore,
\[
\text{Var}(\hat{A}) \geq \frac{2(2n-1)\sigma^2}{n(n+1)}
\]
\[
\text{Var}(\hat{B}) \geq \frac{12\sigma^2}{n(n^2-1)}
\]