Minimum Variance Unbiased Estimator (III)

Complete Sufficient Statistic

1. Completeness

<u>Def</u> A family of distributions $\{\mathbb{P}_\theta : \theta \in \Lambda\}$ is <u>complete</u> if the condition

$$\mathbb{E}_Y[\varphi(Y)] = 0 \qquad \forall \theta \in \Lambda$$

implies

$$\mathbb{P}_\theta[\varphi(Y) = 0] = 1 \qquad \forall \theta \in \Lambda,$$

for any function φ.

<u>Interpretation</u>

Think of inner products:

Let $\underline{\varphi}$ and $\underline{f}$ be two vectors with $\underline{f} \geq 0$.

Then

$$\underline{\varphi}^T \underline{f} = 0 \implies \underline{\varphi} = 0, \text{ because } \underline{f} > 0.$$

Now, think of

$$\mathbb{E}_Y[\varphi(Y)] = \int \varphi(y) f_Y(y;\theta)\, dy$$
$$\simeq \underline{\varphi}^T \underline{f}_Y.$$

So $\underline{\varphi}^T \underline{f}_Y = 0 \implies \underline{\varphi} = 0$, because $\underline{f}_Y > 0$.

Example

(completeness of binomial)

Let $f_Y(y;\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$

Then $$\mathbb{E}_Y[\varphi(Y)] = \sum_{y=0}^{n} \binom{n}{y} \varphi(y)\, \theta^y (1-\theta)^{n-y}$$
$$= (1-\theta)^n \sum_{y=0}^{n} \underbrace{\binom{n}{y} \varphi(y)}_{\stackrel{\text{def}}{=} a_y} \underbrace{\left(\frac{\theta}{1-\theta}\right)^y}_{\stackrel{\text{def}}{=} x^y}$$

So $\underline{a}^T \underline{x} = 0 \implies \underline{a} = 0$, because $x > 0$.

So the family of binomial distributions is complete.

<u>Def</u> Given $Y \sim \mathbb{P}_\theta$. Define a sufficient statistic $T(Y)$ for θ. If the distribution of $T(Y)$, denoted by $\mathbb{Q}_\theta$, is complete, then T is said to be a <u>complete sufficient statistic</u>.

<u>Properties</u>

(1) "Uniqueness": Let $T(Y)$ be a complete suff stat. If $\tilde{g}(T(Y))$ and $\tilde{\tilde{g}}(T(Y))$ are two unbiased estimators, then $\mathbb{P}\left(\tilde{g}(T(Y)) = \tilde{\tilde{g}}(T(Y))\right) = 1$.

Pf: $$\mathbb{E}_{T(Y)}\left[\tilde{g}(T(Y)) - \tilde{\tilde{g}}(T(Y))\right]$$
$$= \mathbb{E}_{T(Y)}\left[\tilde{g}(T(Y))\right] - \mathbb{E}_{T(Y)}\left[\tilde{\tilde{g}}(T(Y))\right]$$
$$= g(\theta) - g(\theta) = 0.$$

So by completeness $\mathbb{P}\left(\tilde{g}(T(Y)) = \tilde{\tilde{g}}(T(Y))\right) = 1$.

(2) "MVUE": Let $T(Y)$ be a complete suff stat, If $\tilde{g}(T(Y))$ is an unbiased estimator, then $\tilde{g}(T(Y))$ must be an MVUE.

Pf: By Rao-Blackwell, if $\hat{g}(Y)$ is an unbiased estimator, we can always find another estimator based on $T(Y)$

$$\tilde{g}(T(Y)) = \mathbb{E}_{Y|T(Y)}[\hat{g}(Y)].$$

Since $T(Y)$ is complete, $\tilde{g}(T(Y))$ is unique. So it must be MVUE.

<u>General Procedures</u> (to obtain MVUE):

(1) Find a complete suff. stat. $T(Y)$.

(2) Find an unbiased estimator $\hat{g}(Y)$.

(3) Use Rao-Blackwell to define

$$\tilde{g}(T(Y)) = \mathbb{E}_{Y|T(Y)}[\hat{g}(Y)].$$

Then, $\tilde{g}(T(Y))$ is MVUE.

Alternatively, we can also

(1) Find a complete suff stat $T(Y)$

(2) Find an estimator that only depend on $T(Y)$ and not Y. Call it $\tilde{g}(T(Y))$.

(3) Show that $\tilde{g}(T(Y))$ is unbiased.

Then $\tilde{g}(T(Y))$ is MVUE.

2. Exponential Family

<u>Def</u> A family of distributions $\{\mathbb{P}_\theta : \theta \in \Lambda\}$ is said to be an exponential family if

$$f_Y(y;\theta) = C(\theta)\exp\left(\sum_{l=1}^{m} Q_l(\theta)T_l(y)\right)h(y)$$

for some functions C, Q_l, T_l, h.

<u>Remark</u>: Many important distributions are exponential family, including Gaussian, binomial, Poisson, Laplacian, geometric, ... etc.

<u>Theorem</u> (Completeness of Exponential Family)
(See Poor IV.C.3 for proof)
Suppose that $\Lambda \subseteq \mathbb{R}^m$ and $\Gamma = \mathbb{R}^n$. Suppose that

$$f_Y(y;\theta) = C(\theta)\exp\left(\sum_{l=1}^{m} \theta_l T_l(y)\right)h(y).$$

Then $T(y) = [T_1(y), \dots, T_m(y)]$ is a complete suff stat if Λ contains a m-dimensional rectangle.

<u>Example</u>

Let $Y_k = \mu S_k + N_k$, $k = 1, \dots, n$,
$N_k \sim N(0, \sigma^2)$.

Find MVUE of μ.

Solution:

Step 1: Find complete suff stat $T(Y)$.

$$f_{\underline{Y}}(\underline{y};\theta) = \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_k-\mu s_k)^2}{2\sigma^2}\right)$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k-\mu s_k)^2\right)$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(\frac{-1}{2\sigma^2}\left[\sum_{k=1}^{n} y_k^2 - 2\mu\sum_{k=1}^{n} y_k s_k + \mu^2\sum_{k=1}^{n} s_k^2\right]\right)$$

$$= \underbrace{\frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(\frac{-\mu^2}{2\sigma^2}\sum_{k=1}^{n} s_k^2\right)}_{C(\theta)} \cdot \underbrace{\exp\left(\frac{\mu}{\sigma^2}\sum_{k=1}^{n} y_k s_k\right)}_{\theta_1 \quad T_1(\underline{y})} \underbrace{\exp\left(\frac{-1}{2\sigma^2}\sum_{k=1}^{n} y_k^2\right)}_{h(\underline{y})}$$

Therefore, $T_1(\underline{y}) = \sum_{k=1}^{n} y_k s_k = \underline{y}^T \underline{s}$ is a complete suff stat because μ is contained in $\mathbb{R}$, which of course contains a rectangle.

Step 2 Define an estimator of μ from $T_1(y)$.

Since $T_1(y) = \sum_{k=1}^{n} y_k s_k$, we have

$$\mathbb{E}[T_1(Y)] = \left(\sum_{k=1}^{n} s_k^2\right)\mu \qquad \left(\text{note: } \mathbb{E}[Y_k] = \mu s_k\right)$$

So we define

$$\hat{g}(T(Y)) = \frac{T_1(Y)}{\|\underline{s}\|^2}.$$

Step 3 : Check whether $\hat{g}(T_1(Y))$ is unbiased.

$$\mathbb{E}[\hat{g}(T_1(Y))] = \frac{1}{\|\underline{s}\|^2}\mathbb{E}[T_1(Y)]$$
$$= \frac{1}{\|\underline{s}\|^2}\left(\mu \|\underline{s}\|^2\right)$$
$$= \mu.$$

So we conclude that the estimator

$$\hat{g}(T(Y)) = \frac{\sum_{k=1}^{n} s_k Y_k}{\|\underline{s}\|^2}$$

is MVUE.