# Lecture 14

## Performance Analysis of Detection

### Large Deviation ~~Ana~~ Theory (II)

1. Error Bound on $P_F$ and $P_M$

$$P_F(\delta) = \int_{\ell(y)>\eta} f_0(y)\,dy + \int_{\ell(y)=\eta} \gamma f_0(y)\,dy + \cancelto{0}{\int_{\ell(y)<\eta} 0\, f_0(y)\,dy}$$

$$\leq \int_{\ell(y)\geq\eta} f_0(y)\,dy, \quad \text{because } \gamma \leq 1.$$

$$= \mathbb{P}(\ell(Y) \geq \eta).$$

Similarly, we can show that

$$P_M(\delta) \leq \mathbb{P}(\ell(Y) \leq \eta).$$

By large deviation theorem, we have

$$P_F(\delta) \leq \frac{\mathbb{E}[e^{s\ell(y)} \mid H_0]}{e^{s\eta}}$$

<u>Def</u> $\mathbb{E}_0[\ell(y)] \overset{\text{def}}{=} \mathbb{E}[\ell(y) \mid H_0]$

$\mathbb{E}_1[\ell(y)] \overset{\text{def}}{=} \mathbb{E}[\ell(y) \mid H_1]$

Our goal is to study $\mathbb{E}[e^{s\ell(y)} \mid H_0]$ and $\mathbb{E}_0[\ell(y)]$.

2. Properties of $\mathbb{E}_0[\ell(y)]$.

By definition of $\mathbb{E}_0[\ell(y)]$, we can show that

$$\mathbb{E}_0[\ell(y)] = \mathbb{E}[\ell(y) \mid H_0]$$
$$= \int \ell(y) f_0(y)\, dy$$
$$= \int \ln \frac{f_1(y)}{f_0(y)} f_0(y)\, dy.$$

Similarly,

$$\mathbb{E}_1[\ell(y)] = \int \ln \frac{f_1(y)}{f_0(y)} f_1(y)\, dy.$$

<u>Def</u>. Kullback-Leibner Divergence

$$D(f_0 \| f_1) = \int \ln \frac{f_0(y)}{f_1(y)} f_0(y)\, dy.$$

Then,

$$\mathbb{E}_0[\ell(y)] = -D(f_0 \| f_1) \leq 0$$

$$\mathbb{E}_1[\ell(y)] = D(f_1 \| f_0) \geq 0.$$

<u>How to show $D(p \| q) \geq 0$ ?</u>

Just need to show $\forall \sum p_i = 1$, $\sum q_i = 1$,

$$-\sum p_i \log p_i \leq -\sum p_i \log q_i.$$

Since $\log x \leq x - 1$, we have

$$-\sum_i p_i \log \frac{q_i}{p_i} \geq \sum p_i \left(\frac{q_i}{p_i} - 1\right) = -\sum_i q_i + \sum_i p_i = -1 + 1 = 0.$$

We can also show that

$$\ln \mathbb{E}_0\left[e^{s\ell(y)}\right] = \ln \mathbb{E}_0\left[e^{s\ln\frac{f_1(y)}{f_0(y)}}\right]$$
$$= \ln\left[\int e^{s\ln\frac{f_1(y)}{f_0(y)}} f_0(y)\,dy\right]$$
$$= \ln\left[\int e^{\ln\frac{f_1^s(y)}{f_0^s(y)}} f_0(y)\,dy\right]$$
$$= \ln\left[\int f_1^{(s)}(y)\, f_0^{(1-s)}(y)\,dy\right] \overset{\text{def}}{=} \mu_0(s).$$

Similarly,

$$\ln \mathbb{E}_1\left[e^{s\ell(y)}\right] = \ln \mathbb{E}_1\left[e^{s\ln\frac{f_1(y)}{f_0(y)}}\right]$$
$$= \ln \int f_1^{(s+1)}(y)\, f_0^{-s}(y)\,dy \overset{\text{def}}{=} \mu_1(s).$$

By changing the variables $s' = s+1$, we can show that $\cancel{\mu_1(s') = \ln \int f_1^{s'}(y)\, f_0^{(1-s')}(y)\,dy = \mu_0(s')}$,

~~So $\mu_1(s) = \mu_0(s)$~~

$$\mu_0(s+1) = \ln \int f_1^{(s+1)}(y)\, f_0^{-s}(y)\,dy = \mu_1(s).$$

Hence

$$P_F(\delta) \le \frac{\mathbb{E}_0\left[e^{s\ell(y)}\right]}{e^{s\eta}} = e^{-(s\eta - \mu_0(s))}, \quad \forall s \ge 0.$$

$$P_M(\delta) \le \frac{\mathbb{E}_1\left[e^{s\ell(y)}\right]}{e^{s\eta}} = e^{-(s\eta - \mu_1(s))} \quad \forall s \le 0$$
$$= e^{-(s\eta - \mu_0(s+1))} \quad \because \mathbb{P}(\ell(y) \le \eta \mid H_1)$$
$$= e^{-((s'-1)\eta - \mu_0(s'))}$$
$$= e^{\eta}\, e^{-(s'\eta - \mu_0(s'))}, \quad \forall s' \le 1.$$

Because

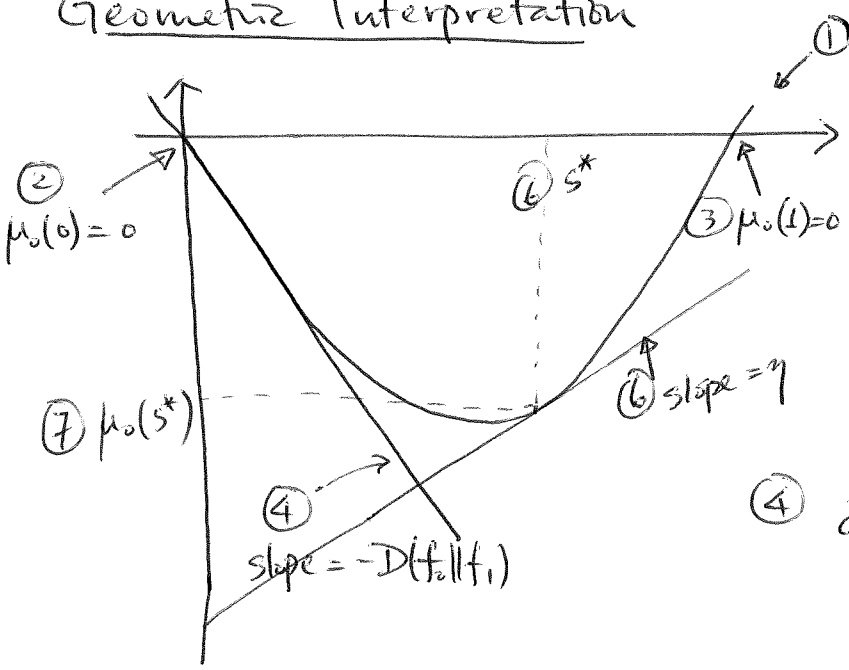
$$P_F(\delta) \le e^{-(s\eta - \mu_0(s))} \quad \forall s \ge 0$$

$$P_M(\delta) \le e^{\eta} e^{-(s\eta - \mu_0(s))}, \quad \forall \cancel{s \ge 1}\ s \le 1,$$

($s$ and $s'$ are dummy variables)

the performance is characterized by $g(s) = s\eta - \mu_0(s)$.

## Geometric Interpretation




Properties of $\mu_0$:

① $\mu_0$ is convex

② $\mu_0(0) = \ln \int f_1^0(y) f_0^1(y)\, dy$
$= \ln \int f_0(y)\, dy = 0$

③ $\mu_0(1) = 0$.

④ $\frac{d}{ds}\mu_0(s) = \frac{d}{ds} \ln \mathbb{E}_0[e^{s\ell(y)}]$
$= \dfrac{\mathbb{E}_0[\ell(y) e^{s\ell(y)}]}{\mathbb{E}_0[e^{s\ell(y)}]}$

$\frac{d}{ds}\mu_0(s)\Big|_{s=0} = \mathbb{E}_0[\ell(y)]$
$= -D(f_0 \| f_1)$.

⑤ $\frac{d}{ds}\mu_1(s)\Big|_{s=0} = D(f_1 \| f_0)$.

⑥ $g(s)$ is maximized at $g'(s) = 0$

i.e. $\frac{dg(s)}{ds} = \eta - \mu_0'(s) = 0$

$\Rightarrow \mu_0'(s) = \eta$

So $s^*$ is maximized ~~at~~ of $g$ ~~at~~ when slope of $\mu_0(s^*) = \eta$

⑦ At $s^*$, $\mu_0$ achieves $\mu_0(s^*)$.

At $s^*$, $g$ achieves $s^*\eta - \mu_0(s^*)$.