Lecture 12
Performance Analysis of Signal Detection
Central Limit Theorem

1. Convergence in distribution

**Def**

A sequence of distributions with CDF $F_1, \ldots, F_n$ is said to converge to another distribution $F$ if $F_n(x) \to F(x)$ at all continuous points $x$ of $F$. Write as $F_n \to F$.

**Def**

A sequence of random variables $Y_1, \ldots, Y_n$ is said to converge to $Y$ in distribution if $F_n \to F$. Write as $Y_n \overset{d}{\to} Y$, or $Y_n \to F$.

**Example**

The notation $Y_n \overset{d}{\to} N(0,1)$ means that the distribution of $Y_n$ is converging to $N(0,1)$. Note that $Y_n \overset{d}{\to} Y$ does not mean that $Y$ is becoming $Y$. But it only means that the distributions are becoming closer.

**Remark:** $Y_n \overset{P}{\to} Y \Rightarrow Y_n \overset{d}{\to} Y$, but the converse is not true.
2. Central Limit Theorem

Theorem 12.1

Let \( X_1, \ldots, X_n \) be iid random variables with 
\[
E[X_k] = \mu \quad \text{and} \quad \text{Var}(X_k) = \sigma^2 < \infty.
\]
Then
\[
\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2),
\]
where \( \bar{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k \).

Proof: It is sufficient to prove that
\[
\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0,1).
\]

Let \( Z_n = \sqrt{n}(\bar{X}_n - \mu) \). The moment generating function of \( Z_n \) is
\[
M_{Z_n}(s) \overset{\text{def}}{=} \mathbb{E}[e^{sZ_n}]
\]
\[
= \mathbb{E} \left[ e^{s \sqrt{n}(\bar{X}_n - \mu)} \right]
\]
\[
= \frac{n}{\sigma \sqrt{n}} \mathbb{E} \left[ e^{\frac{s^2}{\sigma^2 n}(X_k - \mu)} \right]
\]
Taylor expansion \( \overset{\text{Taylor}}{\approx} \)
\[
\overset{\text{Taylor}}{\approx} \frac{n}{\sigma \sqrt{n}} \mathbb{E} \left[ 1 + \frac{3}{\sigma^2 n}(X_k - \mu) + \frac{s^2}{2 \sigma^2 n} (X_k - \mu)^2 + O \left( \frac{(X_k - \mu)^3}{\sigma^3 n \sqrt{n}} \right) \right]
\]
Assume \( \mathbb{E}[X_k^3] < \infty \)
\[
\xrightarrow{n \to \infty} \frac{n}{\sigma \sqrt{n}} \prod_{k=1}^{n} \left( 1 + 0 + \frac{s^2}{2n} \right)
\]
\[
= \left( 1 + \frac{s^2}{2n} \right)^n \xrightarrow{(a)} e^{\frac{s^2}{2}} \quad \text{as} \; n \to \infty.
\]
Since $M_{2n}(s) \rightarrow \frac{s^2}{2}$, we have $Z_n \xrightarrow{d} N(0, 1)$.

To prove (a), we let $y_n = \left(1 + \frac{s^2}{2n}\right)^n$.

Then, $\ln y_n = n \ln(1 + \frac{s^2}{2n})$.

By Taylor expansion,

$$
\ln(1 + x_0) = \ln(1) + \left(\frac{\partial}{\partial x} \ln x\right)_{x=1} x_0 + \frac{1}{2!} \left(\frac{\partial^2}{\partial x^2} \ln x\right)_{x=1} x_0^2 + O(x_0^3).
$$

$$
= x_0 - \frac{x_0^2}{2}.
$$

So $\ln y_n = n \ln(1 + \frac{s^2}{2n}) = n \left(\frac{s^2}{2n} - \frac{s^4}{4n^2}\right)

= \frac{s^2}{2} - \frac{s^4}{4n} \xrightarrow{n \to \infty} \frac{s^2}{2}.

So $y_n \xrightarrow{d} e^{\frac{s^2}{2}}$.

3. Delta Method

**Theorem 12.2**

If $\sqrt{n} \left( T_n - \theta \right) \xrightarrow{d} N(0, \tau^2)$, then

$$\sqrt{n} \left( f(T_n) - f(\theta) \right) \xrightarrow{d} N(0, \tau^2(f'(\theta))^2),$$

provided $f'(\theta)$ exists.

**Pf:** By Taylor expansion,

$$f(T_n) = f(\theta) + (T_n - \theta)f'(\theta) + O((T_n - \theta)^2)$$

$$\Rightarrow \sqrt{n} \left( f(T_n) - f(\theta) \right) = \sqrt{n}(T_n - \theta)f'(\theta).$$

Note that $\sqrt{n}(T_n - \theta)f'(\theta) \xrightarrow{d} N(0, \tau^2(f'(\theta))^2)$.
4. Limitation of Central Limit Theorem

Recall that our analysis question is to study

\[ P \left( \sum_{i=1}^{n} X_i \geq \eta \right) \]

Let's go back to Central Limit Theorem, which can be stated as

\[ \lim_{n \to \infty} P \left[ \left( \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n} \sigma} \right) \leq \varepsilon \right] = \Phi(\varepsilon) \]

\[ \iff \lim_{n \to \infty} P \left( \frac{\sum_{i=1}^{n} X_i - n\mu + n \sigma \varepsilon}{\sqrt{n} \sigma} \right) = \Phi(\varepsilon) \]

think it as "\eta".

\[ \iff \lim_{n \to \infty} P \left( \frac{1}{n} \sum_{i=1}^{n} X_i \leq \mu + \frac{\sigma \varepsilon}{\sqrt{n}} \right) = \Phi(\varepsilon) \]

As \( n \to \infty \), \( \frac{\sigma \varepsilon}{\sqrt{n}} \to 0 \). Therefore, the deviation that Central Limit Theorem can handle is "small deviation'.

\[ \mu \quad \mu + \frac{\sigma \varepsilon}{\sqrt{n}} \]

the deviation must be small in order to get Gaussian approximation

For large deviations, CLT does not work.