Lecture 12

Performance Analysis of Signal Detection

Central Limit Theorem

1. Convergence in distribution

Def

A sequence of distributions with CDF $F_1 \dots F_n$ is said to converge to another distribution F if $F_n(x) \longrightarrow F(x)$ at all continuous points x of F. Write as $F_n \longrightarrow F$.

Def

A sequence of random variables $Y_1 \dots Y_n$ is said to converge to Y in distribution if $F_n \longrightarrow F$. Write as $Y_n \xrightarrow{d} Y$, or $Y_n \xrightarrow{d} F$.

Example

The notation $Y_n \xrightarrow{d} N(0,1)$ means that the distribution of Y_n is converging to $N(0,1)$.

Note that $Y_n \xrightarrow{d} Y$ does not mean that Y is becoming Y. But it only means that the distributions are becoming closer.

Remark: $Y_n \xrightarrow{P} Y \implies Y_n \xrightarrow{d} Y$, but the converse is not true.

2. Central Limit Theorem

<u>Theorem</u> 12.1

Let $X_1 \dots X_n$ be iid random variables with $\mathbb{E}[X_k] = \mu$ and $Var(X_k) = \sigma^2 < \infty$. Then

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2).$$

where $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$.

<u>Pf</u>: It is sufficient to prove that

$$\sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right) \xrightarrow{d} N(0,1).$$

Let $Z_n = \sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right)$. The moment generating function of Z_n is

$$M_{Z_n}(s) \overset{def}{=} \mathbb{E}\left[e^{sZ_n}\right]$$

$$= \mathbb{E}\left[e^{s\sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right)}\right]$$

$$= \prod_{k=1}^{n} \mathbb{E}\left[e^{\frac{s}{\sigma\sqrt{n}}(X_k - \mu)}\right]$$

Taylor expansion

$$= \prod_{k=1}^{n} \mathbb{E}\left[1 + \frac{s}{\sigma\sqrt{n}}(X_k - \mu) + \frac{s^2}{2\sigma^2 n}(X_k - \mu)^2 + O\left(\frac{(X_k - \mu)^3}{\sigma^3 n \sqrt{n}}\right)\right]$$

Assume $\mathbb{E}[X_k^3] < \infty$

$$\overset{n\to\infty}{=} \prod_{k=1}^{n} \left(1 + 0 + \frac{s^2}{2n}\right)$$

$$= \left(1 + \frac{s^2}{2n}\right)^n \xrightarrow[(a)]{} e^{\frac{s^2}{2}} \text{ as } n \to \infty.$$

Since $M_{Z_n}(s) \longrightarrow e^{\frac{s^2}{2}}$, we have $Z_n \xrightarrow{d} N(0,1)$.

To prove (a), we let $y_n = \left(1+\frac{s^2}{2n}\right)^n$.

Then, $\ln y_n = n \ln\left(1+\frac{s^2}{2n}\right)$.

By Taylor expansion,

$$\ln(1+x_0) = \ln(1) + \underbrace{\left(\frac{\partial}{\partial x}\ln x\right)_{x=1}}_{=\frac{1}{x}} x_0 + \frac{1}{2!}\underbrace{\left(\frac{\partial^2}{\partial x^2}\ln x\right)_{x=1}}_{\frac{-1}{x^2}} x_0^2 + O(x_0^3)$$

$$= x_0 - \frac{x_0^2}{2}.$$

So $\ln y_n = n\ln\left(1+\frac{s^2}{2n}\right) = n\left(\frac{s^2}{2n} - \frac{s^4}{4n^2}\right)$

$$= \frac{s^2}{2} - \frac{s^4}{4n} \xrightarrow{n\to\infty} \frac{s^2}{2}.$$

So $y_n \longrightarrow e^{\frac{s^2}{2}}$. ▨

3. Delta Method

<u>Theorem</u> 12.2

> If $\sqrt{n}(T_n - \theta) \xrightarrow{d} N(0,\tau^2)$, then
> $\sqrt{n}(f(T_n) - f(\theta)) \xrightarrow{d} N(0, \tau^2(f'(\theta))^2)$,
> provided $f'(\theta)$ exists.

<u>Pf</u>: By Taylor expansion,

$$f(T_n) = f(\theta) + (T_n-\theta)f'(\theta) + o((T_n-\theta)^2)$$

$$\Rightarrow \sqrt{n}(f(T_n) - f(\theta)) = \sqrt{n}(T_n-\theta)f'(\theta).$$

Note that $\sqrt{n}(T_n-\theta)f'(\theta) \xrightarrow{d} N(0, \tau^2(f'(\theta))^2)$. ▨

4. Limitation of Central Limit Theorem

Recall that our analysis question is to study

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i \geq \eta \right). \tag{11.3}$$

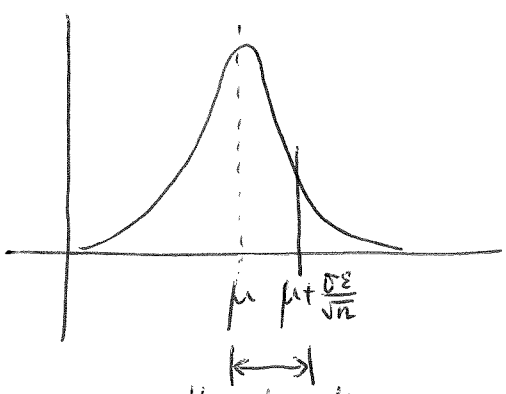
Let's go back to Central Limit Theorem, which can be stated as

$$\lim_{n\to\infty} \mathbb{P}\left[\left(\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\,\sigma} \right) \leq \varepsilon \right] = \Phi(\varepsilon).$$

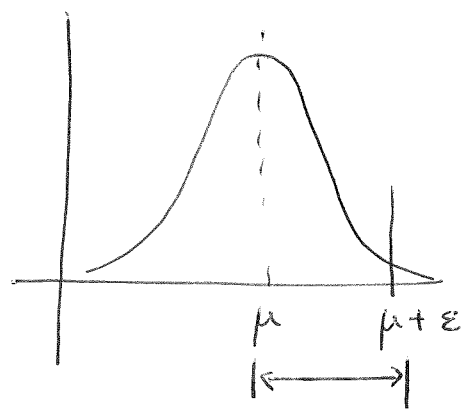
$$\Longleftrightarrow \lim_{n\to\infty} \mathbb{P}\left(\sum_{i=1}^{n} X_i \leq \underbrace{n\mu + \sqrt{n}\,\sigma\varepsilon}_{\text{think it as "}\eta\text{"}} \right) = \Phi(\varepsilon)$$

$$\Longleftrightarrow \lim_{n\to\infty} \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \leq \mu + \frac{\sigma\varepsilon}{\sqrt{n}} \right) = \Phi(\varepsilon).$$

As $n\to\infty$, $\frac{\sigma\varepsilon}{\sqrt{n}} \longrightarrow 0$. Therefore, the deviation that Central Limit Theorem can handle is "small deviation".



the deviation must be small in order to get Gaussian approximation



For large deviations, CLT does not work.