Lecture 11

Performance Analysis of Signal Detection

— Law of Large Numbers

1. Probability Bounds for Pf and Pm

Consider a general detection rule using randomization

\[ \delta(y) = \begin{cases} 1 & \text{if } \ell(y) \geq \eta, \\ 0 & \text{if } \ell(y) < \eta, \end{cases} \]

where \( \ell(y) \) is the log-likelihood ratio.

Given \( \delta(y) \), we can show that

\[ \Pr = \int_{\ell(y) > \eta} f_0(y) dy + \int_{\ell(y) = \eta} R f_0(y) dy \]

\[ \leq \int_{\ell(y) > \eta} f_0(y) dy, \quad \text{because } R \leq 1 \]

\[ = \Pr(\ell(y) \geq \eta \mid H_0). \quad \text{(11.1)} \]

Similarly, we can show that

\[ P_m = \int_{\ell(y) < \eta} f_1(y) dy + \int_{\ell(y) = \eta} (1-R) f_1(y) dy \]

\[ \leq \int_{\ell(y) < \eta} f_1(y) dy \]

\[ = \Pr(\ell(y) < \eta \mid H_1). \]
Now, assume that the observation is a vector \( y \):

\[
y = [y_1, \ldots, y_n]^T.
\]

then,

\[
\int_{\ell(y) \geq \eta} f_0(y) dy = \int_{\ell(y) \geq \eta} f_0(y_i \ldots y_n) dy_i \ldots dyn
\]

\[
iid = \int_{\ell(y) \geq \eta} \prod_{i=1}^n f_0(y_i) dy_i \ldots dyn (11.2)
\]

Unfortunately, (11.2) involves multivariate integrations, and is extremely difficult to compute. To overcome this difficulty, one solution is to derive accurate upper bounds for \( Pf \) and \( Pf_i \):

\[
\ell(y) = \log \frac{f_i(y)}{f_0(y)} = \ln \frac{\prod_{i=1}^n f_i(y_i)}{\prod_{i=1}^n f_0(y_i)} = \sum_{i=1}^n \ell_i(y_i),
\]

where \( \ell_i(y_i) = \ln \frac{f_i(y_i)}{f_0(y_i)} \).

Define \( X_i = \ell_i(y_i) \), then computing \( Pf \) is equivalent to computing (see (11.1))

\[
P(r \sum_{i=1}^n X_i \geq \eta) (11.3)
\]

**Goal**: Understand various ways to obtain upper bounds of (11.3).
2. Markov and Chebyshev Inequality

**Theorem 11.1 (Markov)**

\[
\forall X > 0, \text{ and } \forall \varepsilon > 0, \\
\Pr(X > \varepsilon) \leq \frac{\mathbb{E}[X]}{\varepsilon}
\]

**Proof**:

\[\varepsilon \Pr(X > \varepsilon) = \varepsilon \int_{\varepsilon}^{\infty} f_X(x) \, dx \leq \int_{\varepsilon}^{\infty} x f_X(x) \, dx \leq \int_{0}^{\infty} x f_X(x) \, dx = \mathbb{E}[X].\]

**Theorem 11.2 (Chebyshev)**

Let \( X \) be a random variable such that

\[\mathbb{E}[X] = \mu \text{ and } \operatorname{Var}(X) < \infty, \text{ then } \forall \varepsilon > 0 \]

\[\Pr(|X - \mu| > \varepsilon) \leq \frac{\operatorname{Var}(X)}{\varepsilon^2}\]

**Proof**:

\[\Pr(|X - \mu| > \varepsilon) = \Pr((X - \mu)^2 > \varepsilon^2) \leq \frac{\mathbb{E}[(X - \mu)^2]}{\varepsilon^2} = \frac{\operatorname{Var}(X)}{\varepsilon^2}.\]

3. Weak Law of Large Number

**Corollary 11.3**

Let \( X_1, \ldots, X_n \) be i.i.d random variables with

\[\mathbb{E}[X_k] = \mu \text{ and } \operatorname{Var}(X_k) = \sigma^2. \]

Then if

\[Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k, \text{ then } \Pr(|Y_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{n \varepsilon^2}, \]

\( \forall \varepsilon > 0 \).
\textbf{Pf.} \quad \text{If} \quad \left| Y_n - \mu \right| > \varepsilon \quad \Rightarrow \quad \frac{\mathbb{E}[(Y_n - \mu)^2]}{\varepsilon^2} \\
(\text{By Chebyshev}) \\
\mathbb{E}[(Y_n - \mu)^2] = \text{Var}(Y_n) \\
= \text{Var}\left( \frac{1}{n} \sum_{k=1}^{n} X_k \right) \\
= \frac{1}{n^2} \sum_{k=1}^{n} \text{Var}(X_k) = \frac{1}{n^2} \cdot n \sigma^2 = \frac{\sigma^2}{n}. \\
Therefore, \\
\mathbb{P}\left( \left| Y_n - \mu \right| > \varepsilon \right) \leq \frac{\sigma^2}{n \varepsilon^2}. \quad \square

The interpretation of Corollary 11.3 is important. The corollary says that we have a sequence of iid random variables \( X_1 \ldots X_n \), the mean \( Y_n \) will stay around the mean of \( X_i \). In particular,

\[ \lim_{n \to \infty} \mathbb{P}\left( \left| Y_n - \mu \right| > \varepsilon \right) \leq \lim_{n \to \infty} \frac{\sigma^2}{n \varepsilon^2} = 0. \]

This result is known as the weak law of large number.

\textbf{Example}

Consider a unit square containing an arbitrary shape \( \mathcal{S} \). Let \( X_1 \ldots X_n \) be a sequence of iid Bernoulli random variables with probability \( p = \text{Area of } \mathcal{S} \). Then, if we let \( Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k \),

we can show that \( \mathbb{E}[Y_n] = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[X_k] = \frac{np}{n} = p \), and \( \text{Var}(Y_n) = \frac{1}{n^2} \sum_{k=1}^{n} \text{Var}(X_k) = \frac{1}{n^2} \cdot np(1-p) = \frac{p(1-p)}{n} \). Therefore,

\[ \mathbb{P}\left( \left| Y_n - p \right| > \varepsilon \right) \leq \frac{p(1-p)}{n \varepsilon^2} \to 0 \text{ as } n \to \infty. \]

So we can throw arbitrary \( n \) points to the unit square to compute area \( \mathcal{S} \).
**Definition** We say that a sequence of random variables $Y_n$ converges in probability to $\mu$, denoted as $Y_n \xrightarrow{p} \mu$

if $\lim_{n \to \infty} \Pr(|Y_n - \mu| > \varepsilon) = 0$.

**Corollary 11.4**

If $Y_n \xrightarrow{p} \mu$, then $f(Y_n) \xrightarrow{p} f(\mu)$ for any function $f$ that is continuous at $\mu$.

**Example** Let $X_1, \ldots, X_n$ be iid Poisson$(\lambda)$.

Then if $\overline{X} = \frac{1}{n} \sum_{k=1}^{n} X_k$, and $\overline{X} \xrightarrow{p} \lambda$, then

$$e^{-\overline{X}} \xrightarrow{p} e^{-\lambda}$$

**Proof:** Since $f$ is continuous at $\mu$, we must have that

for all $\varepsilon > 0$, there exists $\delta$ such that

$$|x - \mu| < \delta \implies |f(x) - f(\mu)| < \varepsilon.$$

Therefore,

$$\Pr(|Y_n - \mu| < \delta) \leq \Pr(|f(Y_n) - f(\mu)| < \varepsilon),$$

because $|Y_n - \mu| < \delta$ is a subset of $|f(Y_n) - f(\mu)| < \varepsilon$.

Hence

$$\Pr(|f(Y_n) - f(\mu)| > \varepsilon) \leq \Pr(|Y_n - \mu| > \delta) \rightarrow 0 \text{ as } n \to \infty.$$