Lecture 11

Performance Analysis of Signal Detection — Law of Large Numbers

1. Probability Bounds for P_F and P_M

Consider a general detection rule using randomization

$$\delta(y) = \begin{cases} 1 \\ \gamma \\ 0 \end{cases} \quad \text{if} \quad \ell(y) \begin{matrix} > \\ = \\ < \end{matrix} \eta,$$

where $\ell(y) \overset{\text{def}}{=} \log L(y)$ is the log-likelihood ratio.

Given $\delta(y)$, we can show that

$$P_F = \int_{\ell(y)>\eta} f_0(y)\,dy + \gamma \int_{\ell(y)=\eta} f_0(y)\,dy$$

$$\leq \int_{\ell(y)\geq\eta} f_0(y)\,dy, \quad \text{because } \gamma \leq 1$$

$$= \mathbb{P}(\ell(y) \geq \eta \mid H_0). \qquad (11.1)$$

Similarly, we can show that

$$P_M = \int_{\ell(y)<\eta} f_1(y)\,dy + (1-\gamma) \int_{\ell(y)=\eta} f_1(y)\,dy$$

$$\leq \int_{\ell(y)\leq\eta} f_1(y)\,dy$$

$$= \mathbb{P}(\ell(y) \leq \eta \mid H_1).$$

Now, assume that the observation is a vector $\underline{y}$:

$$\underline{y} = [y_1, \ldots, y_n]^T.$$

then,

$$\int_{\ell(\underline{y}) \geq \eta} f_0(\underline{y})\, d\underline{y} = \int_{\ell(\underline{y}) \geq \eta} f_0(y_1 \ldots y_n)\, dy_1 \ldots dy_n$$

$$\overset{iid}{=} \int_{\ell(\underline{y}) \geq \eta} \prod_{i=1}^{n} f_0(y_i)\, dy_1 \ldots dy_n \quad \text{——} (11.2)$$

Unfortunately, (11.2) involves multivariate integrations, and is extremely difficult to compute. To overcome this difficulty, one solution is to derive accurate <u>upper bounds</u> for P_F and P_M:

$$\ell(\underline{y}) = \log \frac{f_1(\underline{y})}{f_0(\underline{y})} \overset{iid}{=} \ln \frac{\prod_{i=1}^{n} f_1(y_i)}{\prod_{i=1}^{n} f_0(y_i)} = \sum_{i=1}^{n} \ell_i(y_i),$$

where $\ell_i(y_i) = \ln \dfrac{f_1(y_i)}{f_0(y_i)}$.

Define $X_i = \ell_i(y_i)$, then computing P_F is equivalent to computing (See (11.1))

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i \geq \eta \right) \quad \text{———} \quad (11.3)$$

<u>Goal</u>: Understand various ways to obtain upper bounds of (11.3).

2. Markov and Chebyshev Inequality

<u>Theorem</u> 11.1 (Markov)

$$\boxed{\forall X > 0, \text{ and } \forall \varepsilon > 0, \quad \mathbb{P}(X > \varepsilon) \leq \frac{\mathbb{E}[X]}{\varepsilon}}$$

Pf: $$\varepsilon \mathbb{P}(X > \varepsilon) = \varepsilon \int_{\varepsilon}^{\infty} f_X(x)\,dx$$
$$\leq \int_{\varepsilon}^{\infty} x f_X(x)\,dx$$
$$\leq \int_{0}^{\infty} x f_X(x)\,dx = \mathbb{E}[X]. \quad \blacksquare$$

<u>Theorem</u> 11.2 (Chebyshev)

Let X be a random variable such that $\mathbb{E}[X] = \mu$ and $Var(X) < \infty$, then $\forall \varepsilon > 0$

$$\boxed{\mathbb{P}(|X-\mu| > \varepsilon) \leq \frac{Var(X)}{\varepsilon^2}}$$

Pf: $$\mathbb{P}(|X-\mu| > \varepsilon) = \mathbb{P}((X-\mu)^2 > \varepsilon^2)$$
$$\underset{\text{Markov Inequality}}{\leq} \frac{\mathbb{E}[(X-\mu)^2]}{\varepsilon^2} = \frac{Var(X)}{\varepsilon^2}. \quad \blacksquare$$

3. Weak Law of Large Number

<u>Corollary</u> 11.3

Let $X_1 \dots X_n$ be iid random variables with $\mathbb{E}[X_k] = \mu$ and $Var(X_k) = \sigma^2$. ~~then~~ If

$$Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k, \text{ then } \mathbb{P}(|Y_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2},$$

$\forall \varepsilon > 0$.

Pf: $\mathbb{P}(|Y_n-\mu|>\varepsilon) \leq \dfrac{\mathbb{E}[(Y_n-\mu)^2]}{\varepsilon^2}$ (By chebyshev)

$$\begin{aligned}\mathbb{E}[(Y_n-\mu)^2] &= Var(Y_n)\\ &= Var\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right)\\ &= \frac{1}{n^2}\sum_{k=1}^{n} Var(X_k) = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}.\end{aligned}$$

Therefore,

$$\mathbb{P}(|Y_n-\mu|>\varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}. \qquad \blacksquare$$

The interpretation of Corollary 11.3 is important. The Corollary says if we have a sequence of iid random variables $X_1 \dots X_n$, the mean Y_n will stay around the mean of X_1. In particular,

$$\lim_{n\to\infty} \mathbb{P}(|Y_n-\mu|>\varepsilon) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0.$$

This result is known as the <u>weak</u> law of large number.

<u>Example</u>

Consider a unit square containing an arbitrary shape Ω. Let $X_1 \dots X_n$ be a sequence of iid Bernoulli random variables with probability p = Area of Ω. Then if we let

$$Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k,$$

we can show that $\mathbb{E}[Y_n] = \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}[X_k] = \frac{np}{n} = p$, and $Var(Y_n) = \frac{1}{n^2}\sum_{k=1}^{n} Var(X_k) = \frac{1}{n^2} np(1-p) = \frac{p(1-p)}{n}$. Therefore,

$$\mathbb{P}(|Y_n-p|>\varepsilon) \leq \frac{p(1-p)}{n\varepsilon^2} \to 0 \text{ as } n\to\infty.$$

So we can throw arbitrary n points to the unit square to compute area Ω.

Def We say that a sequence of random variables Y_n converges <u>in probability</u> to μ, denoted as

$$Y_n \xrightarrow{P} \mu$$

if $\lim_{n\to\infty} \mathbb{P}(|Y_n - \mu| > \varepsilon) = 0$.

Corollary 11.4

$$\boxed{\text{If } Y_n \xrightarrow{P} \mu, \text{ then } f(Y_n) \xrightarrow{P} f(\mu) \text{ for any function } f \text{ that is continuous at } \mu.}$$

Example Let $X_1 \ldots X_n$ be iid Poisson(λ). Then if $\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k$, and $\bar{X} \xrightarrow{P} \lambda$, then

$$e^{-\bar{X}} \xrightarrow{P} e^{-\lambda}.$$

Pf: Since f is continuous at μ, we must have that $\forall \varepsilon > 0, \exists \delta$ s.t.

$$|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \varepsilon.$$

Therefore,

$$\mathbb{P}(|Y_n - \mu| < \delta) \leq \mathbb{P}(|f(Y_n) - f(\mu)| < \varepsilon),$$

because "$|Y_n - \mu| < \delta$" is a subset of "$|f(Y_n) - f(\mu)| < \varepsilon$".

Hence

$$\mathbb{P}(|f(Y_n) - f(\mu)| > \varepsilon) \leq \mathbb{P}(|Y_n - \mu| > \delta)$$

$$\longrightarrow 0 \text{ as } n \to \infty. \quad \blacksquare$$