Lecture 10

Composite Hypothesis Test (III)
(Locally Most Powerful Test)

1. Locally Most Powerful Test (LMP)

When UMP does not exist; one can use LMP.

To understand LMP, we consider

$$H_0 : Y \sim f_{Y|\Theta}(y|\theta_0) \quad , \quad \theta = \theta_0$$
$$H_1 : Y \sim f_{Y|\Theta}(y|\theta) \quad , \quad \theta \in \Lambda_1 ,$$

and <u>assume</u> that

$$\Lambda_1 = \{\theta : \theta \gtrsim \theta_0\}$$ (that is, θ is very close to θ_0)

Then, by first order approximation, we have

$$\begin{aligned} P_D(\delta;\theta) &= \int_\Gamma \delta(y) f_{Y|\Theta}(y|\theta)\, dy \\ &= \int \delta(y) \Big[f_{Y|\Theta}(y|\theta_0) + \frac{d}{d\theta} f_{Y|\Theta}(y|\theta)\Big|_{\theta=\theta_0}(\theta-\theta_0) \\ &\qquad + o\big((\theta-\theta_0)^2\big) \Big]\, dy \\ &= \underbrace{\int \delta(y) f_{Y|\Theta}(y|\theta_0)\, dy}_{\alpha} \\ &\qquad + (\theta-\theta_0)\int \delta(y)\frac{d}{d\theta} f_{Y|\Theta}(y|\theta)\Big|_{\theta=\theta_0} dy + O\big((\theta-\theta_0)^2\big) \end{aligned}$$

Therefore, to maximize $P_D(\delta;\theta)$, one can just maximize $\int \delta(y) \frac{d}{d\theta}\Big|_{\theta_0} dy$ subject to $\int \delta(y) f_{Y|\Theta}(y|\theta_0)\, dy$ $= \alpha$.

In this case, the test becomes

$$\underset{\delta_{LMP}(y)}{\underset{\uparrow}{\delta(y)}} = \begin{cases} 1 \\ \gamma \\ 0 \end{cases} ; \quad \text{if} \quad \frac{\frac{d}{d\theta} f_{Y|\Theta}(y|\theta)\big|_{\theta=\theta_0}}{f_{Y|\Theta}(y|\theta_0)} \begin{matrix} > \\ = \\ < \end{matrix} \tau .$$

<u>Example</u>

$$H_0 : Y \sim \frac{1}{2}\exp(-|y|)$$
$$H_1 : Y \sim \frac{1}{2}\exp(-|y-\theta|), \quad \theta > 0$$

Find LMP Test.

$$\frac{d}{d\theta} f_{Y|\Theta}(y|\theta) = \begin{cases} \cancel{\tfrac{1}{2}\exp(-y)\exp}\ \frac{1}{2} e^{-y} e^{\theta} , & y > \theta \\ -\frac{1}{2} e^{y} e^{-\theta} , & y < \theta . \end{cases}$$

When $\theta = 0$,

$$\frac{d}{d\theta} f_{Y|\Theta}(y|\theta)\Big|_{\theta=0} = \begin{cases} \frac{1}{2} e^{-y} ; & y > 0 \\ -\frac{1}{2} e^{y} ; & y < 0 \end{cases}$$

$$= \frac{\text{sign}(y)}{2} e^{-|y|} .$$

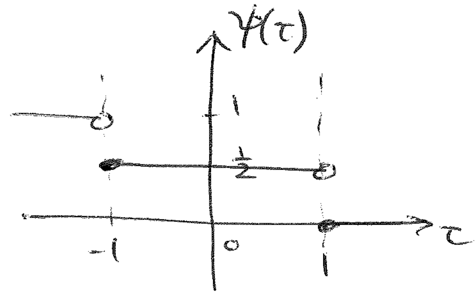
So

$$\delta_{LMP}(y) = \begin{cases} 1 \\ \gamma \\ 0 \end{cases} ; \quad \frac{\frac{\text{sign}(y)}{2} e^{-|y|}}{\frac{1}{2} e^{-|y|}} \begin{matrix} > \\ = \\ < \end{matrix} \tau$$

$$= \text{sign}(y) \begin{matrix} > \\ = \\ < \end{matrix} \tau .$$

Let $\varphi(\tau) = \int_{\text{sign}(y) > \tau} f_{\theta_0}(y)\,dy$, then,

$$\psi(\tau) = \begin{cases} 1 & ; \quad \tau < -1 \\ \frac{1}{2} & ; \quad -1 \le \tau < 1 \\ 0 & ; \quad \tau \geq 1 \end{cases}$$



So if $\frac{1}{2} < \alpha < 1$, $\gamma = \dfrac{\alpha - \frac{1}{2}}{1 - \frac{1}{2}} = 2\alpha - 1$.

if $0 < \alpha < \frac{1}{2}$, $\gamma = \dfrac{\alpha - 0}{\frac{1}{2} - 0} = 2\alpha$.

Finally, we have

$$\delta_{LMP}(y) = \begin{cases} 1 \\ 2\alpha - 1 \\ 0 \end{cases} ; \quad \text{if } y \gtreqless 0 \quad , \frac{1}{2} < \alpha < 1$$

or

$$\delta_{LMP}(y) = \begin{cases} 2\alpha \\ 0 \end{cases} ; \quad \text{if } \begin{matrix} y > 0 \\ y < 0 \end{matrix} \quad , 0 < \alpha < \frac{1}{2}.$$

2. Generalize Likelihood Ratio Test (GLRT)

Another method for composite hypothesis testing when UMP does not exist is the GLRT. The idea is to construct a two-step approach:

(1) Compute

$$\hat{\theta}_0 = \underset{\theta \in \Lambda_0}{\arg\max}\, f_{Y|\theta}(y|\theta)$$

$$\hat{\theta}_1 = \underset{\theta \in \Lambda_1}{\arg\max}\, f_{Y|\theta}(y|\theta)$$

(maximum likelihood! (TBD) in Part II of course

(2) Implicitly assume $H_0 : Y \sim f_{\hat{\theta}_0}$
$H_1 : Y \sim f_{\hat{\theta}_1}$.

Check

$$\delta_G(y) = \begin{cases} 1 \\ \gamma \\ 0 \end{cases} ; \quad \frac{f_{\hat{\theta}_1}(y)}{f_{\hat{\theta}_0}(y)} \gtreqless \tau.$$

Example

$$H_0: \; Y \sim \mathcal{N}(0, \sigma^2)$$

$$H_1: \; Y \sim \mathcal{N}(\theta, \sigma^2), \qquad \theta \neq 0.$$

Find GLRT rule.

Solution: $\hat{\theta}_0 = \underset{\theta \in \{0\}}{\arg\max} \; f_{\theta_0}(y) = 0.$

$$\hat{\theta}_1 = \underset{\theta \in \{\theta \neq 0\}}{\arg\max} \; \mathcal{N}(y; \theta, \sigma^2)$$

$$= \underset{\theta}{\arg\max} \; \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\theta)^2}{2\sigma^2}\right)$$

$$= \underset{\theta}{\arg\min} \; (y-\theta)^2 = y.$$

So $f_{\hat{\theta}_0}(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\cancel{\theta})^2}{2\sigma^2}}$

$$f_{\hat{\theta}_1}(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-y)^2}{2\sigma^2}\right) = \frac{1}{\sqrt{2\pi\sigma^2}}.$$

So $L_{GLR}(y) = \dfrac{f_{\hat{\theta}_1}(y)}{f_{\hat{\theta}_0}(y)} = e^{\frac{y^2}{2\sigma^2}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$

$$\Rightarrow \quad |y| \underset{H_1}{\overset{H_0}{\lessgtr}} \sigma\sqrt{2\ln\eta}.$$