Homework 3

Q1 Let $Y = A + V$,

where $V \sim \text{uniform}[\frac{-1}{2}, \frac{1}{2}]$, and A is unknown.

Let $H_0 : A = 0$

$H_1 : 0 < A \le 1$.

(a) Determine whether the test has the monotone likelihood property.

(b) Does UMP exist?

Solution

(a) We can show that

$$L(y) = \frac{f_1(y)}{f_0(y)} = \frac{\text{Uniform}[A-\frac{1}{2}, A+\frac{1}{2}]}{\text{Uniform}[\frac{-1}{2}, \frac{1}{2}]}$$

$$= \begin{cases} 0 & , \quad \frac{-1}{2} \le y < \frac{-1}{2} + A \\ 1 & , \quad \frac{-1}{2} + A \le y \le \frac{1}{2} \\ \infty & , \quad \frac{1}{2} < y < \frac{1}{2} + A. \end{cases}$$

Since $L(y)$ is not strictly increasing, the monotone likelihood test fails.

(b) Since $L(y)$ is discrete valued, we need randomization. To determine the randomization parameter, we note that

$$L(y) > \eta \iff y > \tau$$

for some ~~so~~ τ, because $L(y)$ is increasing.

Therefore, we have

$$P_F = \int_{y > \tau} f_0(y)\,dy = \int_{\tau}^{\frac{1}{2}} 1\,dy = \frac{1}{2} - \tau$$

Setting $P_F = \alpha$, we have $\tau = \cancel{0}\ \frac{1}{2} - \alpha$.

Thus, the NP decision rule is

$$Y \gtreqless \frac{1}{2} - \alpha .$$

Since the rule does not depend on A, it is UMP.

Q2 Let $\{Y_1 \ldots Y_N\}$ be iid Poisson(λ), i.e.

$$\mathbb{P}(Y_k = n) = \frac{\lambda^n}{n!} e^{-\lambda}.$$

(a) Show that $S = \sum_{k=1}^{N} Y_k$ is a suff stat for estimating λ.

(b) Let $H_0 : \lambda = \lambda_0$,
$H_1 : \lambda = \lambda_1$,

find the likelihood ratio test.

(c) does UMP exist?

<u>Solution</u>

(a)

$$f_{\underline{Y}}(\underline{y} ; \lambda) = \prod_{k=1}^{N} f_{Y_k}(y_k ; \lambda)$$

$$= \prod_{k=1}^{N} \left[\frac{\lambda^{y_k}}{y_k!} \exp(-\lambda) \right]$$

$$= \frac{\exp(-N\lambda)}{\prod_{k=1}^{N} y_k!} \exp\left(\cancel{x} \ln\left(\cancel{y} \lambda^{\sum_{k=1}^{N} y_k} \right) \right)$$

$$= \frac{\exp(-N\lambda)}{\prod_{k=1}^{N} y_k!} \exp\left(\left(\sum_{k=1}^{N} y_k \right) \ln \lambda \right)$$

Let $S(\underline{Y}) = \sum_{k=1}^{N} Y_k$, we obtain a suff statistics.

(b) First, note that for any sequence of Poisson random variables, $Y_1 \dots Y_N$, the sum $S(\underline{Y}) = Y_1 + \dots + Y_N$ is also a Poisson random vaiable with parameter $N\lambda$.

Therefore

$$L(s) = \frac{f_S(s;\lambda_1)}{f_S(s;\lambda_0)}$$

$$= \frac{(N\lambda_1)^s \exp(-N\lambda_1)/s!}{(N\lambda_0)^s \exp(-N\lambda_0)/s!}$$

$$= \left(\frac{\lambda_1}{\lambda_0}\right)^s \exp(N(\lambda_0 - \lambda_1)).$$

Taking ln, this gives

$$s \ln\left(\frac{\lambda_1}{\lambda_0}\right) \gtrless N(\lambda_1 - \lambda_0) + \ln\tau .$$

Since $\lambda_1 > \lambda_0$, we have $\ln\left(\frac{\lambda_1}{\lambda_0}\right) > 0$

$$S \gtrless \frac{N(\lambda_1 - \lambda_0)}{\ln(\lambda_1/\lambda_0)} + \frac{\ln\tau}{\ln(\lambda_1/\lambda_0)} \qquad (*)$$

(c) Clearly, when λ_1 and λ_0 are specified, S is independent of λ. So ~~UMP~~ the test in (*) is UMP.

Q3 Let Y be a random variable with

$$f_Y(y;\theta) = \frac{1}{2}\exp(-|y-\theta|).$$

Consider $H_0: \theta = 0$

$H_1: \theta > 0$

Determine whether UMP exists.

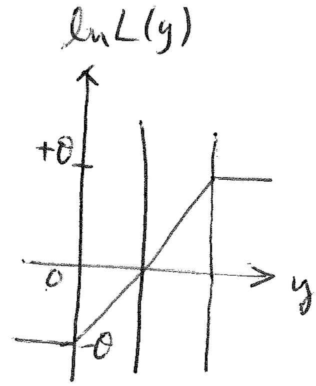
<u>Solution</u>

$$L(y) = \frac{f_1(y)}{f_0(y)} = \frac{\frac{1}{2}\exp(-|y-\theta|)}{\frac{1}{2}\exp(-|y|)}$$

$$= \exp(|y| - |y-\theta|).$$

So $\ln L(y) = |y| - |y-\theta|$

$$= \begin{cases} -\theta & ; \quad y \le 0 \\ 2y-\theta & ; \quad 0 \le y \le \theta \\ \theta & ; \quad y > \theta . \end{cases}$$



$\ln L(y)$ is not strictly increasing. So likelihood ratio test does yield a define answer.

However, since $\ln L(y)$ is increasing,

$$\ln L(y) > \tau \iff y > \eta \text{ for some } \eta.$$

So $P_F = \int_\eta^\infty f_0(y)\,dy = \int_\eta^\infty \frac{1}{2}\exp(-|y|)\,dy = \frac{1}{2}\exp(-\eta).$

$\Rightarrow \quad \alpha = \frac{1}{2}\exp(-\eta)$

$\Rightarrow$

$\eta = -\ln(2\alpha)$. indep of θ. So UMP.

Q4 $Y \sim N(0, v)$

Consider $H_0: v = v_0$

$H_1: v > v_0$.

Find δ_G, the decision rule of Generalized likelihood ratio test (GLRT).

Solution:

$$f(y;v) = \cancel{\exp}\frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{y^2}{2v}\right)$$

$$\Rightarrow \ln f(y;v) = \frac{-y^2}{2v} - \frac{1}{2}\ln(2\pi v)$$

$$\Rightarrow \frac{d}{dv}\ln f(y;v) = \frac{1}{2}\left[\frac{y^2}{v^2} - \frac{1}{v}\right] = 0$$

$$\Rightarrow \hat{v} = Y^2 \Rightarrow \hat{v}_1 = \max(Y^2, v_0)$$

max is needed here because Y^2 may or may not be $> v_0$. if $Y^2 < v_0$, then max is achieved at v_0.

Therefore

$$L_G(y) = \frac{f(y;\hat{v}_1)}{f(y;v_0)} = \left(\frac{v_0}{\hat{v}_1}\right)^{1/2} \exp\left(\frac{Y^2}{2v_0} - \frac{Y^2}{2\hat{v}_1}\right)$$

$$\Rightarrow \ln L_G(y) = \frac{1}{2}\left[\frac{Y^2}{v_0} - \frac{Y^2}{\hat{v}_1} - \ln(\hat{v}_1/v_0)\right]$$

$$\Rightarrow \quad = \begin{cases} \frac{1}{2}\left[\frac{\hat{v}_1}{v_0} - \frac{\hat{v}_1}{\hat{v}_1} - \ln(\hat{v}_1/v_0)\right] & \hat{v}_1 = Y^2 \\ 0 & \hat{v}_1 = v_0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left[\frac{\hat{v}_1}{v_0} - 1 - \ln(\hat{v}_1/v_0)\right) & \hat{v}_1 = Y^2 \\ 0 & \hat{v}_1 = v_0 \end{cases}$$

So $\ln L_G(y) \gtrless \eta \Rightarrow \frac{1}{2}\left[\frac{\hat{v}_1}{v_0} - 1 - \ln\left(\frac{\hat{v}_1}{v_0}\right)\right] \gtrless \eta$.

Let $D\left(\frac{V_1}{V_0}\right) = \frac{1}{2}\left[\frac{V_1}{V_0} - 1 - \ln\left(\frac{V_1}{V_0}\right)\right]$.

then one can show that D^{-1} exists.

Therefore

$$D(\hat{V}_1/V_0) \gtrless \eta$$

$$\Rightarrow \quad \hat{V}_1/V_0 \gtrless D^{-1}(\eta) \overset{\text{def}}{=} \tau$$

$$\Rightarrow \quad \max\left(\frac{Y^2}{V_0}, 1\right) \gtrless \tau$$

$$\Rightarrow \quad \max\left(\frac{|Y|}{V_0^{\frac{1}{2}}}, 1\right) \gtrless \tau^{\frac{1}{2}}.$$