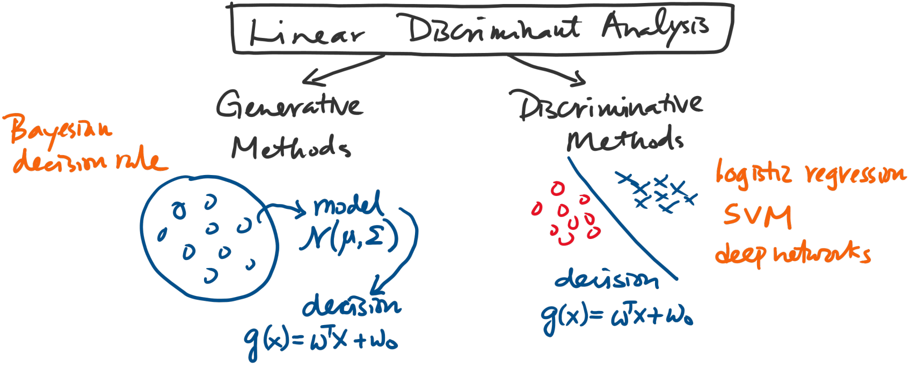
ECE595 / STAT598: Machine Learning I
Lecture 16 Perceptron: Definition and Concept

Spring 2020

Stanley Chan

School of Electrical and Computer Engineering
Purdue University

Overview



- In linear discriminant analysis (LDA), there are generally two types of approaches
- **Generative approach**: Estimate model, then define the classifier
- **Discriminative approach**: Directly define the classifier

Outline

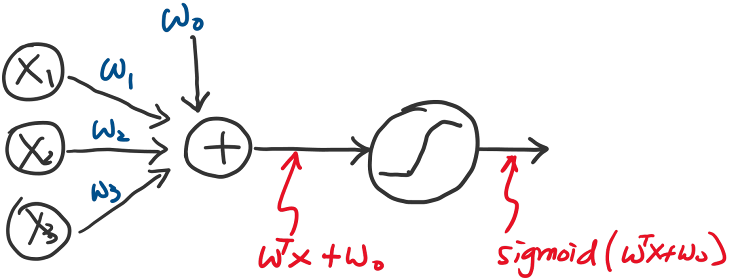
Discriminative Approaches

- Lecture 16 Perceptron 1: Definition and Basic Concepts
- Lecture 17 Perceptron 2: Algorithm and Property

This lecture: Perceptron 1

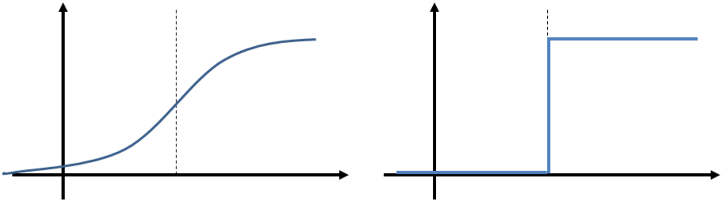
- From Logistic to Perceptron
  - What is Perceptron? Why study it?
  - Perceptron Loss
  - Connection with other losses
- Properties of Perceptron Loss
  - Convexity
  - Comparing with Bayesian Oracle
  - Preview of Perceptron Algorithm

Perceptron as a Single-Layer Network



- Logistic regression: Soft threshold
- Perceptron: Hard threshold

From Logistic to Perceptron



- Logistic regression

$$h(x) = \frac{1}{1 + e^{-a(x - x_0)}}.$$

- Make $a \to \infty$, then $h(x) \to$ step function

$$\lim_{a \to \infty} h(x) = \lim_{a \to \infty} \frac{1}{1 + e^{-a(x - x_0)}}$$
$$= \text{sign}(a(x - x_0)).$$

From Logistic to Perceptron

- Linear regression

$$h_{\boldsymbol{\theta}}(\boldsymbol{x}) = \text{sign}(\boldsymbol{w}^T\boldsymbol{x} + w_0).$$

  - Stage 1: Training the discriminant function

$$g_{\boldsymbol{\theta}}(\boldsymbol{x}) = \boldsymbol{w}^T\boldsymbol{x} + w_0.$$

  - Stage 2: Threshold to make decision

$$h_{\boldsymbol{\theta}}(\boldsymbol{x}) = \text{sign}(g_{\boldsymbol{\theta}}(\boldsymbol{x})).$$

- Logistic regression

$$h_{\boldsymbol{\theta}}(\boldsymbol{x}) = \frac{1}{1 + e^{-(\boldsymbol{w}^T\boldsymbol{x} + w_0)}}.$$

- Perceptron algorithm

$$h_{\boldsymbol{\theta}}(\boldsymbol{x}) = \text{sign}(\boldsymbol{w}^T\boldsymbol{x} + w_0).$$

How to Define Perceptron Loss Function

- Logistic regression

$$J(\boldsymbol{\theta}) = \sum_{n=1}^{N} -\Big\{ y_n \log h_{\boldsymbol{\theta}}(\boldsymbol{x}_n) + (1 - y_n) \log(1 - h_{\boldsymbol{\theta}}(\boldsymbol{x}_n)) \Big\}$$

  - Okay if $h_{\boldsymbol{\theta}}(\boldsymbol{x}_n)$ is soft-decision.
  - Not okay if $h_{\boldsymbol{\theta}}(\boldsymbol{x}_n)$ is binary: Either all fit or none fit.
- "Candidate" perceptron loss function

$$J(\boldsymbol{\theta}) = \sum_{n=1}^{N} \max\Big\{ -y_n h_{\boldsymbol{\theta}}(\boldsymbol{x}_n), 0 \Big\}.$$

  - Does not have the log-term
  - Will not run into $\pm\infty$

Understanding the Perceptron Loss function

- "Candidate" perceptron loss function

$$J_{\text{hard}}(\boldsymbol{\theta}) = \sum_{n=1}^{N} \max\Big\{ -y_n h_{\boldsymbol{\theta}}(\boldsymbol{x}_n), 0 \Big\}.$$

- $h_{\boldsymbol{\theta}}(\boldsymbol{x}_n) = \text{sign}(\boldsymbol{w}^T \boldsymbol{x}_n + w_0)$ is either $+1$ or -1.
- If the decision is correct, then must have
  - $h_{\boldsymbol{\theta}}(\boldsymbol{x}_n) = +1$ and $y_n = +1$
  - $h_{\boldsymbol{\theta}}(\boldsymbol{x}) = -1$ and $y_n = -1$
  - In both cases, $y_n h_{\boldsymbol{\theta}}(\boldsymbol{x}_n) = +1$
  - So the loss is $\max\{-y_n h_{\boldsymbol{\theta}}(\boldsymbol{x}_n), 0\} = 0$.
- If the decision is wrong, then must have
  - $h_{\boldsymbol{\theta}}(\boldsymbol{x}_n) = +1$ and $y_n = -1$
  - $h_{\boldsymbol{\theta}}(\boldsymbol{x}_n) = -1$ and $y_n = +1$
  - In both cases, $y_n h_{\boldsymbol{\theta}}(\boldsymbol{x}_n) = -1$
  - So the loss is $\max\{-y_n h_{\boldsymbol{\theta}}(\boldsymbol{x}_n), 0\} = 1$.
- $J(\boldsymbol{\theta})$ is not differentiable in $\boldsymbol{\theta}$.

Perceptron Loss function

- Define the **perceptron loss** as

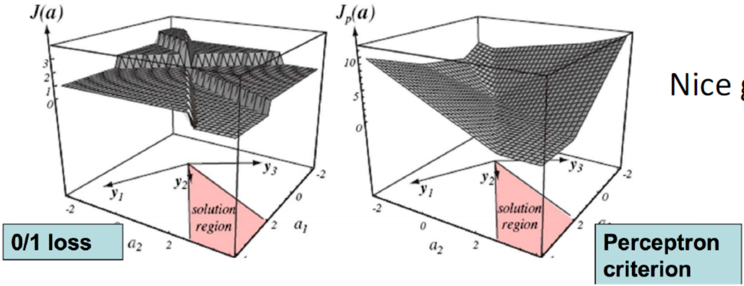
$$J_{\text{soft}}(\boldsymbol{\theta}) = \sum_{n=1}^{N} \max\Big\{ -y_n g_{\boldsymbol{\theta}}(\boldsymbol{x}_n), 0 \Big\}.$$

- $g_{\boldsymbol{\theta}}(\boldsymbol{x}_n) = \boldsymbol{w}^T \boldsymbol{x}_n + w_0$ is either +ve or -ve.
- If the decision is correct, then must have
  - $g_{\boldsymbol{\theta}}(\boldsymbol{x}_n) > 0$ and $y_n = +1$
  - $g_{\boldsymbol{\theta}}(\boldsymbol{x}) < 0$ and $y_n = -1$
  - In both cases, $y_n g_{\boldsymbol{\theta}}(\boldsymbol{x}_n) > 0$
  - So the loss is $\max\{-y_n g_{\boldsymbol{\theta}}(\boldsymbol{x}_n), 0\} = 0$.
- If the decision is wrong, then must have
  - $g_{\boldsymbol{\theta}}(\boldsymbol{x}_n) > 0$ and $y_n = -1$
  - $g_{\boldsymbol{\theta}}(\boldsymbol{x}_n) < 0$ and $y_n = +1$
  - In both cases, $y_n g_{\boldsymbol{\theta}}(\boldsymbol{x}_n) < 0$
  - So the loss is $\max\{-y_n g_{\boldsymbol{\theta}}(\boldsymbol{x}_n), 0\} > 0$.

Comparing the loss function

- **Linear regression**
  - $J(\boldsymbol{\theta}) = \sum_{n=1}^{N} (g_{\boldsymbol{\theta}}(\boldsymbol{x}_n) - y_n)^2$
  - Convex, closed-form solution
  - Usually: Unique global minimizer
- **Logistic regression**
  - $J(\boldsymbol{\theta}) = \sum_{n=1}^{N} -\Big\{ y_n \log h_{\boldsymbol{\theta}}(\boldsymbol{x}_n) + (1 - y_n) \log(1 - h_{\boldsymbol{\theta}}(\boldsymbol{x}_n)) \Big\}$
  - Convex, no closed-form solution
  - Usually: Unique global minimizer
- **Perceptron (Hard)**
  - $J_{\text{hard}}(\boldsymbol{\theta}) = \sum_{n=1}^{N} \max \Big\{ -y_n h_{\boldsymbol{\theta}}(\boldsymbol{x}_n), 0 \Big\}$
  - Not convex, no closed-form solution
  - Usually: Many global minimizers
- **Perceptron (Soft)**
  - $J_{\text{soft}}(\boldsymbol{\theta}) = \sum_{n=1}^{N} \max \Big\{ -y_n g_{\boldsymbol{\theta}}(\boldsymbol{x}_n), 0 \Big\}$
  - Convex, no closed-form solution
  - Usually: Unique global minimizer

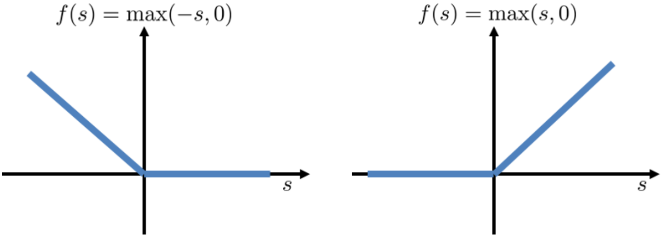
Comparing the loss function



Nice gradient

`https://www.cc.gatech.edu/~bboots3/CS4641-Fall2016/Lectures/Lecture5.pdf`

Perceptron Loss, Hinge Loss and ReLU

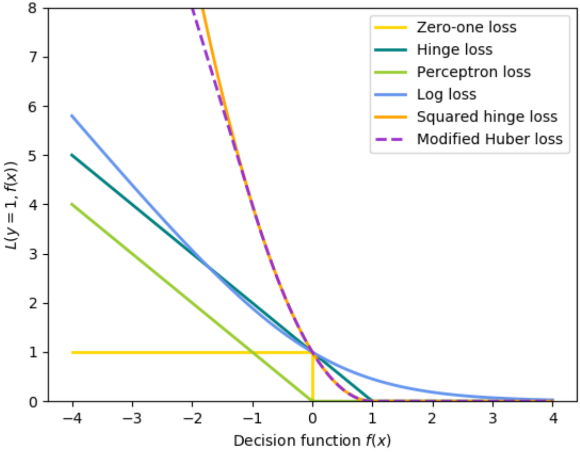


Perceptron Loss　　　　Rectified Linear Unit

- The function $f(s) = \max(-s, 0)$ is called the perceptron loss
- A variant $\max(1 - s, 0)$ is called Hinge Loss
- Another variant $\max(s, 0)$ is called ReLU
- We can prove that the gradient of $f(\boldsymbol{s}) = \max(-\boldsymbol{x}^T\boldsymbol{s}, 0)$ is

$$\nabla_{\boldsymbol{s}} \max(-\boldsymbol{x}^T\boldsymbol{s}, 0) = \begin{cases} -\boldsymbol{x}, & \text{if} \quad \boldsymbol{x}^T\boldsymbol{s} < 0, \\ 0, & \text{if} \quad \boldsymbol{x}^T\boldsymbol{s} \geq 0. \end{cases}$$

Comparing Loss functions



https://scikit-learn.org/dev/auto_examples/linear_model/plot_sgd_loss_functions.html

Outline

Discriminative Approaches

- Lecture 16 Perceptron 1: Definition and Basic Concepts
- Lecture 17 Perceptron 2: Algorithm and Property

This lecture: Perceptron 1

- From Logistic to Perceptron
  - What is Perceptron? Why study it?
  - Perceptron Loss
  - Connection with other losses
- Properties of Perceptron Loss
  - Convexity
  - Comparing with Bayesian Oracle
  - Preview of Perceptron Algorithm

Convexity of Perceptron (Soft) Loss

Let us consider the Perceptron (Soft) Loss

$$J_{\text{soft}}(\boldsymbol{\theta}) = \sum_{n=1}^{N} \max\Big\{ -y_n g_{\boldsymbol{\theta}}(\boldsymbol{x}_n), 0 \Big\}$$

- Is this convex?
- Pick any $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. Pick $\lambda \in [0,1]$.
- We want to show that

$$J(\lambda\boldsymbol{\theta}_1 + (1-\lambda)\boldsymbol{\theta}_2) \leq \lambda J(\boldsymbol{\theta}_1) + (1-\lambda)J(\boldsymbol{\theta}_2)$$

- But notice that

$$\begin{aligned} y_n g_{\boldsymbol{\theta}}(\boldsymbol{x}_n) &= y_n(\boldsymbol{w}^T\boldsymbol{x}_n + w_0) = (y_n\boldsymbol{x}_n)^T\boldsymbol{w} + y_n w_0 \\ &= \begin{bmatrix} y_n\boldsymbol{x}_n^T & y_n \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ w_0 \end{bmatrix} = \boldsymbol{a}^T\boldsymbol{\theta}. \end{aligned}$$

Convexity of Perceptron (Soft) Loss

- Basic fact: If $f(\cdot)$ is convex, then $f(\boldsymbol{A}(\cdot) + \boldsymbol{b})$ is also convex.
- Recognize

$$f(s) = \max\left\{ -s, 0\right\}$$

- So if we can show that $f(s) = \max\{-s, 0\}$ is convex (in s), then

$$f(\boldsymbol{a}^T\boldsymbol{\theta})$$

  is also convex. Put $s = \boldsymbol{a}^T\boldsymbol{\theta}$.
- Let $\lambda \in [0, 1]$, and consider two points $s_1, s_2 \in \text{dom} f$
- Want to show that

$$f(\lambda s_1 + (1-\lambda)s_2) \leq \lambda f(s_1) + (1-\lambda)f(s_2).$$

Convexity of Perceptron (Soft) Loss

- Want to show that

$$f(\lambda s_1 + (1-\lambda)s_2) \leq \lambda f(s_1) + (1-\lambda) f(s_2).$$

- Use the fact that $\max(a+b,0) \leq \max(a,0) + \max(b,0)$
- Equality when $(a > 0$ and $b > 0)$ or $(a < 0$ and $b < 0)$
- Then we can show that

$$\begin{aligned} f(\lambda s_1 + (1-\lambda)s_2) &= \max(-(\lambda s_1 + (1-\lambda)s_2), 0) \\ &\leq \max\Big\{ -\lambda s_1, 0 \Big\} + \max\Big\{ -(1-\lambda)s_2, 0 \Big\} \\ &= \lambda \max(-s_1, 0) + (1-\lambda)\max(-s_2, 0) \\ &= \lambda f(s_1) + (1-\lambda) f(s_2). \end{aligned}$$

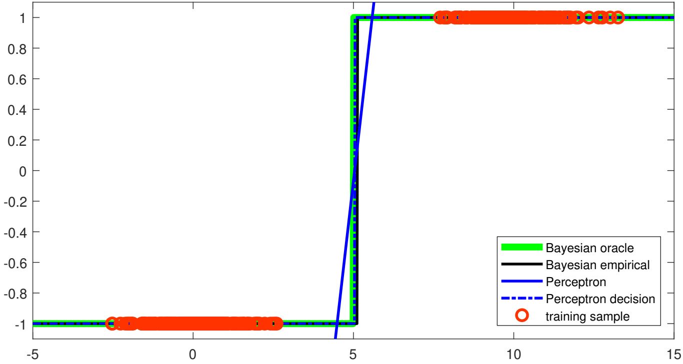
- So the perceptron (soft) loss is convex.
- Therefore, $J_{\text{soft}}(\boldsymbol{\theta})$ is convex in $\boldsymbol{\theta}$.

Implication of Convexity

- You can use CVX to solve the (soft) problem!
- **Existence**: There must exists $\boldsymbol{\theta}^* \in \text{dom}J$ such that $J(\boldsymbol{\theta}^*) \leq J(\boldsymbol{\theta})$ for any $\boldsymbol{\theta} \in \text{dom}J$
- **Uniqueness**: Any local minimizer is also a global minimizer with unique global optimal value.
- **Optimal Value**: If the two classes are linearly separable, then the global minimum is achieved when $J(\boldsymbol{\theta}^*) = 0$
- That means all training samples are classified correctly
- If the two classes are not linearly separable, then you can still get a solution. But $J(\boldsymbol{\theta}^*) > 0$.

Comparing Perceptron and Bayesian Oracle

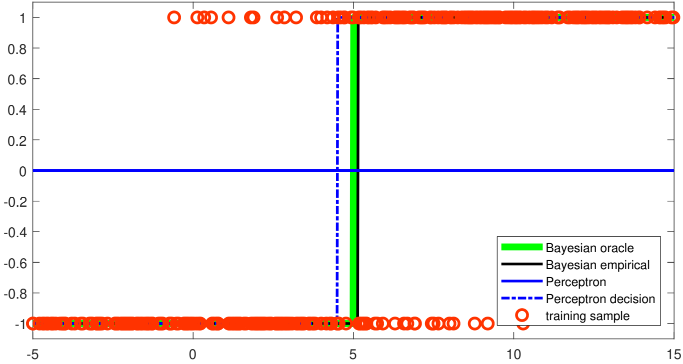
- **Scenario 1**:
- $\mathcal{N}(0, 2)$ with 50 samples and $\mathcal{N}(10, 2)$ with 50 samples.



- When everything is "ideal", perceptron is pretty good.

Comparing Perceptron and Bayesian Oracle

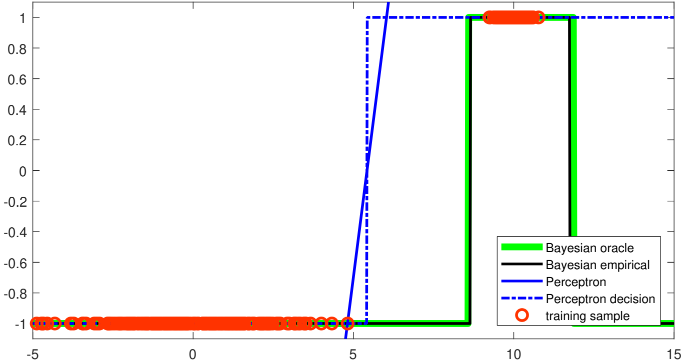
- **Scenario 2**:
- $\mathcal{N}(0, 4)$ with 200 samples and $\mathcal{N}(10, 4)$ with 200 samples.



- Even when datasets are intrinsically overlapping, perceptron is still okay.

Comparing Perceptron and Bayesian Oracle

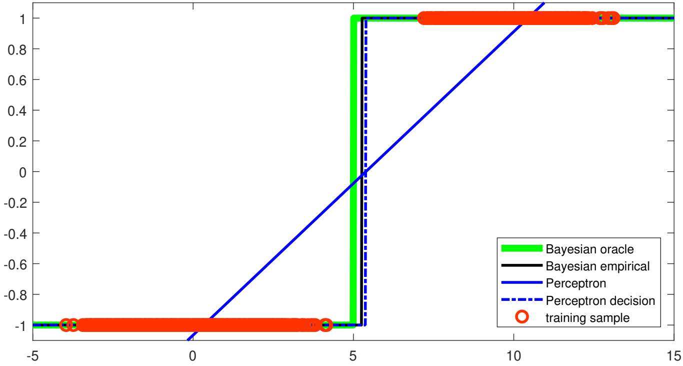
- **Scenario 3**:
- $\mathcal{N}(0,2)$ with 200 samples and $\mathcal{N}(10,0.3)$ with 200 samples.



- When Gaussians have different covariances, the perceptron (as a linear classifier) does not work.

Comparing Perceptron and Bayesian Oracle

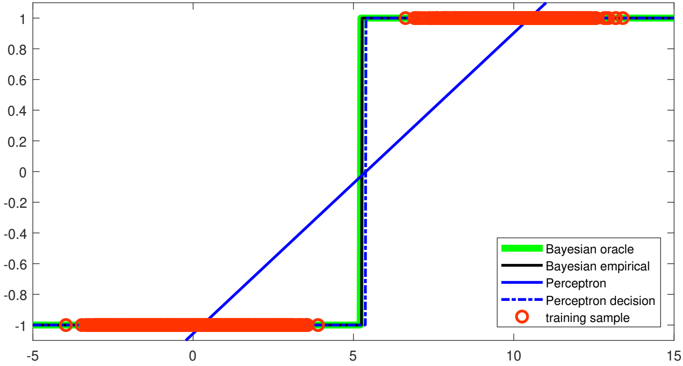
- **Scenario 4**:
- $\mathcal{N}(0,1)$ with 1800 samples and $\mathcal{N}(10,1)$ with 200 samples.



- Number of training samples, in this example, does not seem to affect the algorithm.

Comparing Perceptron and Bayesian Oracle

- **Scenario 5**: 1800 samples and 200 samples.
- $\mathcal{N}(0,1)$ with $\pi_0 = 0.9$ and $\mathcal{N}(10,1)$ with $\pi_1 = 0.1$.



- Intrinsic imbalance between the two distributions does not seem to affect the algorithm.

Perceptron with Hard Loss

- Historically, we have perceptron algorithm way earlier than CVX.
- Before the age of CVX, people solve perceptron using gradient descent.
- Let us be explicit about which loss:

$$J_{\text{hard}}(\boldsymbol{\theta}) = \sum_{j=1}^{N} \max\left\{ -y_j h_{\boldsymbol{\theta}}(\boldsymbol{x}_j), 0 \right\}$$

$$J_{\text{soft}}(\boldsymbol{\theta}) = \sum_{j=1}^{N} \max\left\{ -y_j g_{\boldsymbol{\theta}}(\boldsymbol{x}_j), 0 \right\}$$

- **Goal**: To get a solution for $J_{\text{hard}}(\boldsymbol{\theta})$
- **Approach**: Gradient descent on $J_{\text{soft}}(\boldsymbol{\theta})$

Re-defining the Loss

- **Main idea**: Use the fact that

$$J_{\text{soft}}(\boldsymbol{\theta}) = \sum_{j=1}^{N} \max\Big\{ -y_j g_{\boldsymbol{\theta}}(\boldsymbol{x}_j), 0 \Big\}$$

is the same as this loss function

$$J(\boldsymbol{\theta}) = -\sum_{j \in \mathcal{M}(\boldsymbol{\theta})} y_j g_{\boldsymbol{\theta}}(\boldsymbol{x}_j).$$

- $\mathcal{M}(\boldsymbol{\theta}) \subseteq \{1, \ldots, N\}$ is the set of misclassified samples.
- Run gradient descent on $J(\boldsymbol{\theta})$, but fixing $\mathcal{M}(\boldsymbol{\theta}) \leftarrow \mathcal{M}(\boldsymbol{\theta}^k)$ for iteration k.

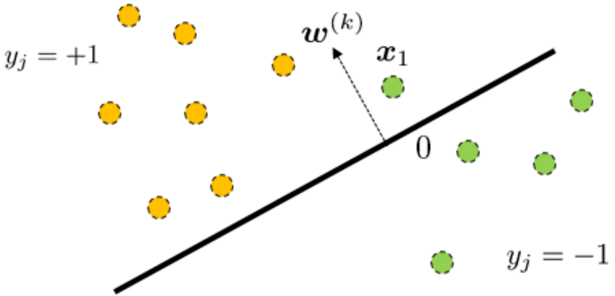
Perceptron Algorithm

- Therefore, the algorithm is
- For $k = 1, 2, \ldots,$
- Update $\mathcal{M}_k = \{j \mid y_j g_{\boldsymbol{\theta}}(\boldsymbol{x}_j) < 0\}$ for $\boldsymbol{\theta} = \boldsymbol{\theta}^{(k)}$.
- Gradient descent

$$\begin{bmatrix} \boldsymbol{w}^{(k+1)} \\ w_0^{(k+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}^{(k)} \\ w_0^{(k)} \end{bmatrix} + \alpha_k \sum_{j \in \mathcal{M}_k} \begin{bmatrix} y_j \boldsymbol{x}_j \\ y_j \end{bmatrix}.$$

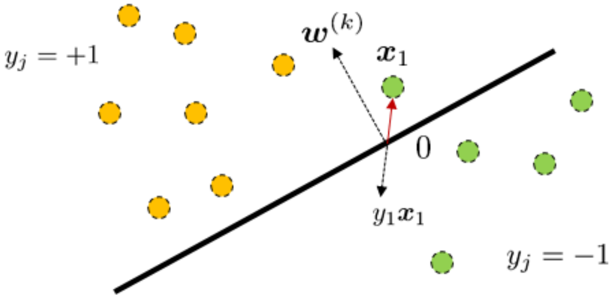
- End For
- The set $\mathcal{M}_k$ can grow or can shrink from $\mathcal{M}_{k-1}$.
- If training samples are linearly separable, then converge. Zero training loss.
- If training samples are not linearly separable, then oscillates.

Updating One Sample



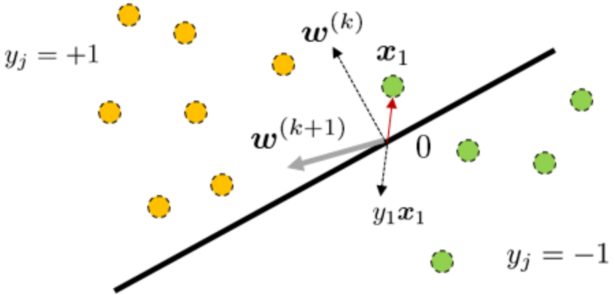
- Initially there is a $\boldsymbol{w}^{(k)}$.
- There is a mis-classifier training sample $\boldsymbol{x}_1$

Updating One Sample



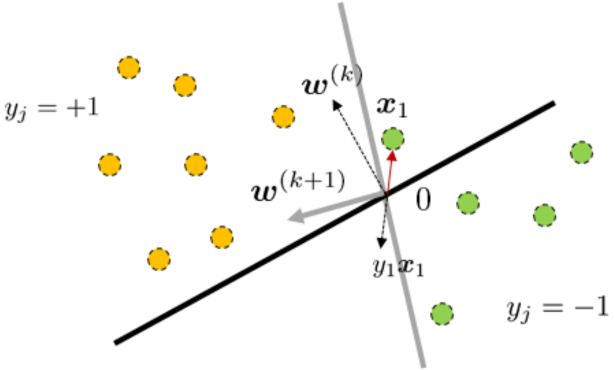
- Perceptron algorithm finds $y_1\boldsymbol{x}_1$
- $y_1\boldsymbol{x}_1$ is in the opposite direction as $\boldsymbol{x}_1$

Updating One Sample



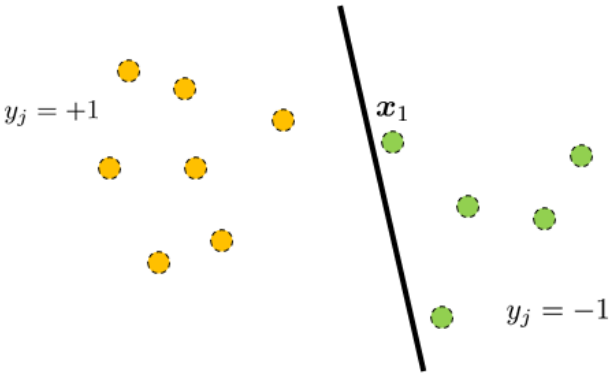
- $\boldsymbol{w}^{(k+1)}$ is a linear combination of $\boldsymbol{w}^{(k)}$ and $y_1\boldsymbol{x}_1$.

Updating One Sample



- $\boldsymbol{w}^{(k+1)}$ gives a new separating hyperplane.

Updating One Sample



- Now you are happy!

Reading List

Perceptron Algorithm

- Abu-Mostafa, Learning from Data, Chapter 1.2
- Duda, Hart, Stork, Pattern Classification, Chapter 5.5
- Cornell CS 4780 Lecture https://www.cs.cornell.edu/courses/cs4780/2018fa/lectures/lecturenote03.html
- UCSD ECE 271B Lecture http://www.svcl.ucsd.edu/courses/ece271B-F09/handouts/perceptron.pdf