Homework 3  
Fall 2015  
(Due: Sep 11, 2015)

Homework is due on Sep 11, 2015 (Friday) at 12:00 noon. Please put your homework in the dropbox located at MSEE 268. No late homework will be accepted.

This homework is difficult. Start early. To do well in this homework you need to be familiar with conditional probabilities, Bayes theorem and Law of total probability. Exercise 5 is taken from 2014 Fall Mid term. Practice this problem well. Exercise 6 is a famous problem called the Three Prisoner’s paradox. Have fun :) 

Exercise 1. (Basic, 10 points)
Let $A$, $B$, $C$ be events with probabilities $P[A] = 0.5$, $P[B] = 0.2$, $P[C] = 0.4$. Find

(a) $P[A \cup B]$ if $A$ and $B$ are independent
(b) $P[A \cup B]$ if $A$ and $B$ are disjoint
(c) $P[A \cup B \cup C]$ if $A$, $B$ and $C$ are independent
(d) $P[A \cup B \cup C]$ if $A$, $B$ and $C$ are pairwise disjoint; Can this happen?

Exercise 2. (Basic, 10 points)
A block of 100 bits is transmitted over a binary communication channel with probability of bit error $p = 10^{-2}$.

(a) If the block has 1 or fewer errors, then the receiver accepts the block. Find the probability that the block is accepted.

(b) If the block has more than 1 error, then the block is retransmitted. What is the probability that 4 blocks are transmitted?

Exercise 3. (Exam Type, 10 points)
A machine makes errors in a certain operation with probability $p$. There are two types of errors. The fraction of errors that are type A is $\alpha$, and type B is $1 - \alpha$.

(a) What is the probability of $k$ errors in $n$ operations?
(b) What is the probability of $k_1$ type A errors in $n$ operations?
(c) What is the probability of $k_2$ type B errors in $n$ operations?
(d) What is the joint probability of $k_1$ type A errors and $k_2$ type B errors in $n$ operations? Hint: There are $\binom{n}{k_1}\binom{n-k_1}{k_2}$ possibilities of having $k_1$ type A errors and $k_2$ type B errors in $n$ operations. (Why?)

Exercise 4. (Exam Type, 15 points)
One of two coins is selected at random and tossed three times. The first coin comes up heads with probability $p_1 = 1/3$ and the second coin with probability $p_2 = 2/3$.

(a) What is the probability that the number of heads is $k = 3$?
(b) Repeat (a) for \( k = 0, 1, 2 \).

(c) Find the probability that coin 1 was tossed given that \( k \) heads were observed, for \( k = 0, 1, 2, 3 \).

(d) In part (c), which coin is more probably when 2 heads have been observed?

**Exercise 5. (Mid Term 2014 Fall, 35 points)**

Consider the following communication channel. A source transmits a string of binary symbols through a noisy communication channel. Each symbol is 0 or 1 with probability \( p \) and \( 1 - p \), respectively, and is received incorrectly with probability \( \varepsilon_0 \) and \( \varepsilon_1 \), respectively. Errors in different symbols transmissions are independent.

Denote \( S \) as the source and \( R \) as the receiver.

(a) What is the probability that a symbol is correctly received? Hint: Find \( P[R = 1 \cap S = 1] \) and \( P[R = 0 \cap S = 0] \).

(b) Find the probability of receiving 1011 conditioned on that 1011 was sent, i.e., \( P[R = 1011 \mid S = 1011] \).

(c) In an effort to improve reliability, each symbol is transmitted three times and the received string is decoded by majority rule. In other words, a 0 (or 1) is transmitted as 000 (or 111, respectively), and it is decoded at the receiver as a 0 (or 1) if and only if the received three-symbol string contains at least two 0s (or 1s, respectively). What is the probability that the symbol is correctly decoded, given that we send a 0?

(d) Suppose that the scheme of part (c) is used. What is the probability that a 0 was sent conditioned on that the string 101 was received?

(e) Suppose the scheme of part (c) is used, and given that a 0 was sent. For what value of \( \varepsilon_0 \) is there an improvement in the probability of correct decoding? Assume that \( \varepsilon_0 \neq 0 \).

**Exercise 6. (Exam Type, 20 points)**

A king announces to release two out of three prisoners (\( A \), \( B \) and \( C \)) and execute the third one, but keeps their identities secret. One of the prisoners, prisoner \( A \), considers asking a friendly guard to tell who is the prisoner other than himself that will be released. However, he hesitates because of the following rationale: Based on his present state, his probability of being released is 2/3. But if he knows the answer, the probability of being released will become 1/2, since there will be two prisoners (including himself) whose fate is unknown and exactly one of the two will be released.

What is wrong with prisoner \( A \)'s argument?

Hint: Let \( A \), \( B \), \( C \) be the events of executing prisoners \( A \), \( B \), \( C \), respectively. Let \( X \) be the event that the guard says that the prisoner \( B \) is released. Determine the probabilities \( P(A) \), \( P(B) \), \( P(C) \), \( P(X \mid A) \), \( P(X \mid B) \) and \( P(X \mid C) \). Hence determine the probability \( P(A^c \mid X) \), which is the probability that prisoner \( A \) is released conditioned on the fact that the guard says prisoner \( B \) is released.