Homework 12
Fall 2015
(Due: Dec 4, 2015)

Homework is due on Dec 4, 2015 (Friday) at 12:00 noon. Please put your homework in the dropbox located at MSEE 268. No late homework will be accepted.

This homework is longer than usual because we have two weeks (including the thanksgiving break). The coverage is the WSS process through LTI system (chapter 10.1-10.2 of the textbook). By completing this homework, I expect you to be familiar with $R_X(\tau)$, $S_X(\omega)$, $R_Y(\tau)$, and $S_Y(\omega)$. Note that ALL questions in this homework are included in the final exam. Therefore, make sure you are fluent about these problems.

Exercise 1. (Basic, 5 points)
Find the autocorrelation function $R_X(\tau)$ corresponding to each of the following power spectral densities:

(a) $\delta(\omega - \omega_0) + \delta(\omega + \omega_0)$
(b) $e^{-\omega^2/2}$
(c) $e^{-|\omega|}$

Exercise 2. (Basic, 5 points)
A WSS process $X(t)$ with autocorrelation function $R_X(\tau) = 1/(1 + \tau^2)$ is passed through an LTI system with impulse response $h(t) = 3\sin(\pi t)/(\pi t)$. Let $Y(t)$ be the system output. Find $S_Y(\omega)$. Sketch $S_Y(\omega)$

Exercise 3. (Basic, 10 points)
A WSS process $X(t)$ with autocorrelation function $R_X(\tau) = e^{-\tau^2/(2\sigma^2)}$ is passed through an LTI system with transfer function $H(\omega) = e^{-\omega^2/(2\sigma^2)}$. Denote the system output by $Y(t)$. Find

(a) $S_{XY}(\omega)$
(b) $R_{XY}(\tau)$
(c) $S_Y(\omega)$
(d) $R_Y(\tau)$

Exercise 4. (Basic, 10 points)
A white noise $X(t)$ with power spectral density $S_X(\omega) = N_0/2$ is applied to a lowpass filter $h(t)$ with

$$H(\omega) = \begin{cases} 1 - \omega^2, & \text{if } |\omega| \leq \pi \\ 0, & \text{otherwise.} \end{cases}$$

Find $E[|Y(t)|^2]$, where $Y(t)$ is the output of the filter.
Exercise 5. (Basic, 10 points)
Let $X(t)$ be a WSS process with correlation function

$$R_X(\tau) = \begin{cases} 1 - |\tau|, & \text{if } -1 \leq \tau \leq 1 \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

It is known that when $X(t)$ is input to a system with transfer function $H(\omega)$, the system output $Y(t)$ has a correlation function

$$R_Y(\tau) = \frac{\sin \pi \tau}{\pi \tau}. \quad (2)$$

Find the transfer function $H(\omega)$.

Exercise 6. (Basic, 10 points)
A white noise $X(t)$ with power spectral density $S_X(\omega) = N_0/2$ is applied to a lowpass filter $h(t)$ with impulse response

$$h(t) = \frac{1}{RC}e^{-t/RC}, \quad t > 0. \quad (3)$$

(a) $S_{XY}(\omega)$
(b) $R_{XY}(\tau)$
(c) $S_Y(\omega)$
(d) $R_Y(\tau)$

Exercise 7. (Exam Type, 10 points)
Consider the system

$$Y(t) = e^{-t} \int_{-\infty}^{t} e^\tau X(\tau)d\tau.$$

Assume that $X(t)$ is zero mean white noise with power spectral density $S_X(\omega) = N_0/2$. Find

(a) $S_{XY}(\omega)$
(b) $R_{XY}(\tau)$
(c) $S_Y(\omega)$
(d) $R_Y(\tau)$

Exercise 8. (Basic, 10 points)
Consider a WSS process $X(t)$ with autocorrelation function

$$R_X(\tau) = \text{sinc}(\pi \tau).$$

The process is sent to an LTI system, with input-output relationship

$$2\frac{d^2}{dt^2}Y(t) + 2\frac{d}{dt}Y(t) + 4Y(t) = 3\frac{d^2}{dt^2}X(t) - 3\frac{d}{dt}X(t) + 6X(t).$$

Find the autocorrelation function $R_Y(\tau)$. 
Exercise 9. (Exam Type, 15 points)
A random process $X(t)$ has zero mean and autocorrelation function $R_X(t, s) = \min(t, s)$. Consider the new process

$$Y(t) = e^t X(e^{-2t}).$$

This process $Y(t)$ is passed through an LTI system, of which the output $Z(t)$ is related to the input $Y(t)$ via

$$\frac{d}{dt}Z(t) + 2Z(t) = \frac{d}{dt}Y(t) + Y(t).$$

(a) Show that $Y(t)$ is wide sense stationary.
(b) Find the autocorrelation function $R_Z(\tau)$.

Exercise 10. (Exam Type, 15 points)
Let $X[n]$ be a white noise with power spectral density $S_X(\omega) = N_0/2$. Let

$$Y[n] = \alpha_0 X[n] + \alpha_1 X[n - 1] + \ldots + \alpha_p X[n - p].$$

(a) Show that for all $|k| > p$, $R_Y[k] = 0$.
(b) Show that for all $|k| \leq p$,

$$R_Y[k] = \frac{N_0}{2} \sum_{i=-\infty}^{\infty} \alpha_i \alpha_{i-k},$$

where $\alpha_i = 0$ for $i > p$ or $i < 0$.
(c) Show that

$$S_Y(\omega) = \frac{N_0}{2} \left( \sum_{i=-\infty}^{\infty} \alpha_i e^{-j\omega i} \right) \left( \sum_{\ell=-\infty}^{\infty} \alpha_\ell e^{j\omega \ell} \right).$$

(d) From (c), what is $H(\omega)$?