Homework 10
Fall 2015
(Thursday, Nov 12, 2015)

Homework is due on Nov 13, 2015 (Friday) at 12:00 noon. Please put your homework in the dropbox located at MSEE 268. No late homework will be accepted.

This homework covers Central Limit Theorem, least squares fitting, and basic random process terminologies. The random process part of this homework is very basic. You should become fluent about finding the mean function $m_X(t)$, the autocorrelation function $R_X(t_1, t_2)$, and the autocovariance function $C_X(t_1, t_2)$.

**Exercise 1. (Basic, 10 points)**

Let $X_1, \ldots, X_n$ be a sequence of iid random variables with mean $E[X_i] = \mu$ and variance $\text{Var}[X_i] = \sigma^2$. The distribution of $X_i$ is, however, unknown. Let $T_n = \sum_{i=1}^{n} X_i$. Use Central Limit Theorem to estimate the probability $P[T_n > 2n\mu]$.

**Exercise 2. (Basic, 10 points)**

Let $X_1, \ldots, X_n$ be a sequence of iid random variables such that $X_i = \pm 1$ with equal probability. Let $Y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$. Prove the Central Limit Theorem for this particular sequence of random variables by showing that

(a) $E[Y_n] = 0$, $\text{Var}[Y_n] = 1$.

(b) The moment generating function of $Y_n$ is $M_{Y_n}(s) \to e^{s^2/2}$ as $n \to \infty$.

**Exercise 3. (Basic, 10 points)**

Given the functions $a(t), b(t)$ and $c(t)$, let

$g(t, 1) = a(t)$
$g(t, 2) = b(t)$
$g(t, 3) = c(t)$.

Let $X(t) = g(t, Z)$, where $Z$ is a discrete random variable with PMF $P[Z = 1] = p_1$, $P[Z = 2] = p_2$, $P[Z = 3] = p_3$. Find $m_X(t)$ and $R_X(t_1, t_2)$ in terms of the $p_1$, $p_2$, $p_3$, $a(t)$, $b(t)$ and $c(t)$.

**Exercise 4. (Basic, 10 points)**

(a) Show that $C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1)m_X(t_2)$.

(b) Show that $C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - m_X(t_1)m_Y(t_2)$.

**Exercise 5. (Exam Type, 15 points)**

Consider the random process

$$X(t) = A \cos(t) + (B + 1) \sin(t),$$


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Instructor: Prof. Stanley H. Chan
(a) Find $m_X(t)$
(b) Find $R_X(t_1, t_2)$
(c) Find $C_X(t_1, t_2)$

Exercise 6. (Basic, 10 points)
Show that $R_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)]$ is a positive semidefinite function in the sense that for any real constants $c_1, \ldots, c_n$ and any times $t_1, \ldots, t_n$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j R_X(t_i, t_j) \geq 0. \quad (1)$$

Hint: Note that $\mathbb{E}[|\sum_{i=1}^{n} c_i X(t_i)|^2] \geq 0.$

Exercise 7. (Basic, 15 points)
Let $X[n]$ be a discrete-time random process with mean function $m_X[n] = \mathbb{E}\{X[n]\}$ and correlation function $R_{X[n, m]} = \mathbb{E}\{X[n]X[m]\}$. Suppose that

$$Y[n] = \sum_{i=-\infty}^{\infty} h[n-i]X[i]. \quad (2)$$

(a) Find $m_Y[n]$ 
(b) Find $R_{XY}[n, m]$ 
(c) Based on (a) and (b), show that $R_{Y[n, m]} = \sum_{\ell=\infty}^{\infty} h[\ell] \left( \sum_{k=-\infty}^{\infty} h[k] R_{X[n-\ell, m-k]} \right).$

Exercise 8. (Exam Type, 10 points)
Let $Z[1], \ldots, Z[n]$ be iid zero-mean random variable with common variance $\sigma^2 = \text{Var}[Z[i]]$ for all $i$. Let

$$X[n] = Z[1] + \ldots Z[n],$$

for $n = 1, 2, \ldots$. Find $R_{X[m, n]}$.

Hint: Show that if $m > n$, then $\mathbb{E}[X[m]X[n]] = n\sigma^2$, whereas if $n > m$, then $\mathbb{E}[X[m]X[n]] = m\sigma^2$.

Exercise 9. (Basic, 10 points)
This is a MATLAB exercise. The datasets are available on the course website.

(a) Download `HW10Data1.mat`. Find the regression coefficients $(a, b, c)$ such that $y = ax^2 + bx + c$. Plot the raw data using `scatter` and overlay the predicted curve using `hold on`.

(b) Download `HW10Data2.mat`. Find the regression coefficients $(a, b)$ such that $y = be^{ax}$. Plot the raw data using `scatter` and overlay the predicted curve using `hold on`. 
