Random Process

A random process is a family of random variables, typically expressed as a sequence. In principle, the sequence can be finite, e.g. \( \{A_1, A_2, A_3\} \), but in practice a random process usually refers to infinite family \( \{A_1, A_2, A_3, \ldots\} \). The need for working with infinite family of random variables arises when we have an indetermined amount of data. For example, sending bits but we do not know how much to send.

Def

A random process \( X(t, \xi) \) is a function of \( t \) indexed by a "random ID" \( \xi \).

Illustration

Each random process \( X(t, \xi) \) is a function in the sample space \( \Omega \). The "ID" is determined by \( \xi \).
Note: $X(t, \xi)$ is a two-dimensional function. Therefore, we can analyze $X(t, \xi)$ from two views:

1. **Fix $t$.** In this case, we can look at the sequence:
   \[
   \begin{align*}
   &X(t, \xi_1) \\
   &X(t, \xi_2) \\
   &\vdots \\
   &X(t, \xi_n)
   \end{align*}
   \]
   This sequence represents a list of $n$ possible realizations of a random variable.

2. **Fix $\xi$.** In this case, the sequence we should look at is
   \[X(t_1, \xi), X(t_2, \xi), \ldots, X(t_k, \xi)\]
   This sequence represents a fixed deterministic function.

Remark: The way I write the two cases above is to emphasize the dimension at which we need to look at.

\[X(t, \xi_1) \uparrow \quad \text{\begin{tikzpicture}
\begin{axis}[
width=\textwidth,
height=6cm,
axis x line=bottom,
axis y line=left,
\]
\addplot[domain=0:10, samples=100,smooth] {sin(x)};
\end{axis}
\end{tikzpicture}\quad \rightarrow t\]

\[\downarrow \quad \text{fix } t, \quad \text{vary } \xi \quad X(t, \xi_2) \uparrow \quad \text{\begin{tikzpicture}
\begin{axis}[
width=\textwidth,
height=6cm,
axis x line=bottom,
axis y line=left,
\]
\addplot[domain=0:10, samples=100,smooth] {sin(x)};
\end{axis}
\end{tikzpicture}\quad \rightarrow t\]

\[\downarrow \quad \text{fix } t, \quad \text{vary } \xi \quad X(t, \xi_n) \uparrow \quad \text{\begin{tikzpicture}
\begin{axis}[
width=\textwidth,
height=6cm,
axis x line=bottom,
axis y line=left,
\]
\addplot[domain=0:10, samples=100,smooth] {sin(x)};
\end{axis}
\end{tikzpicture}\quad \rightarrow t\]
Example (continuous time)

Let \( A \sim \text{Uniform}[0, 1] \). Define \( X(t, \xi) = A(\xi) \cos(2\pi t) \).

If \( A(\xi_1) = 0.5 \), then \( X(t, \xi_1) = 0.5 \cos(2\pi t) \).

If \( A(\xi_2) = 1 \), then \( X(t, \xi_2) = 1 \cos(2\pi t) \).

\[ X(t, \xi_1) \quad \xrightarrow{\text{t}} \quad X(t, \xi_2) \]

Fix \( t = 10 \) (for example),
\[ X(t, \xi) = A(\xi) \cos(2\pi (10)) = A(\xi) \cos(20\pi) \]

\( A(\xi) \cos(2\pi) \) is a random variable, because \( \cos(20\pi) \) is a constant.

\( A(\xi) \sim \text{Uniform}[0, 1] \).

Fix \( \xi \) (for example \( A(\xi) = 0.7 \)),
\[ X(t, \xi) = 0.7 \cos(2\pi t) \]

This is a function in \( t \).

Example (discrete time)

Let \( A \) be a discrete random variable with PMF
\[ P(A = +1) = \frac{1}{2} \]
\[ P(A = -1) = \frac{1}{2} \]

Let \( X(n, \xi) = A(\xi) (-1)^n \).

If \( A = +1 \), then \( X(n, \xi) = (-1)^n \).

If \( A = -1 \), then \( X(n, \xi) = (-1)^{n+1} \).
The sample space contains only two elements
\[ X(n, \xi) = (-1)^n \]
\[ X(n, \xi) = (-1)^{n+1} \]
Therefore,
\[
\mathbb{P}(X(n) = (-1)^n) = \frac{1}{2}
\]
\[
\mathbb{P}(X[n] = (-1)^{n+1}) = \frac{1}{2}
\]
but \[
\mathbb{P}(X(n) = (1, 1, 1, -1, 1, -1, \ldots)) = 0,
\]
because this sequence is outside the sample space.

**Generalization**

Consider a pool of functions \( X_1(t), X_2(t), \ldots, X_n(t) \). Assign a random variable \( N \) with PMF
\[
\mathbb{P}(N = i) = p_i.
\]
Then, the sample space contains \( n \) functions, each with probability \( p_i \) being picked.
Characterization of Random Process

To characterize a single random variable \( X \), we need the PDF \( f_X(x) \). To characterize a pair of random variables \((X, Y)\), we need the joint PDF \( f_{X,Y}(x,y) \). How about a random process? The difficulty comes because a random process is a collection of infinitely many random variables.

Fortunately, Kolmogorov showed that to characterize a random process \( X(t) \), we only need to compute the joint PDF of \( X(t_1), X(t_2), \ldots, X(t_n) \) for a finite number of time instants \( t_1, \ldots, t_n \). That is, as long as we know

\[
f_{X(t_1), \ldots, X(t_n)}(x_1, \ldots, x_n),
\]
we will be able to characterize the entire random process \( X(t) \). But this is still not easy because the joint PDF is a \( n \)-dimensional function.

Our goal: To study a subset of random processes that can be characterized by its "mean" and "variance". This is called stationary process or wide sense stationary process.
Mean Function

**Def** The mean function $M_X(t)$ of a random process $X(t)$ is

$$M_X(t) = E[X(t)]$$

**Example**

Let $A \sim \text{Uniform}(0,1)$, let $X(t) = A \cos(2\pi t)$. Find $M_X(0)$ and $M_X(t)$.

**Sol:**

$$M_X(0) = E[X(0)] = E[A \cos(0)] = E[A] = \frac{1}{2}.$$  

$$M_X(t) = E[X(t)] = E[A \cos(2\pi t)] = \cos(2\pi t) E[A] = \frac{1}{2} \cos(2\pi t).$$

**Example**

Let $\Theta \sim \text{Uniform}(-\pi, \pi)$. Let $X(t) = \cos(\omega t + \Theta)$. Find $M_X(t)$.

**Sol:**

$$M_X(t) = E[\cos(\omega t + \Theta)] = \int_{-\pi}^{\pi} \cos(\omega t + \Theta) \cdot \frac{1}{2\pi} \, d\Theta = 0.$$
Example

\[ X[n] = s^n \quad , \quad S \sim \text{uniform } [0, 1]. \]

Find \( m_X[n] \).

Sol: \( m_X[n] = \mathbb{E}[S^n] \)

\[ = \int_0^1 s^n \, ds = \frac{1}{n+1}. \]

So \( m_X[0] = 1 \)

\( m_X[1] = \frac{1}{2} \)

\( m_X[2] = \frac{1}{3} \)

\[ m_X[n] = \frac{1}{n+1}. \]

\[
\begin{array}{cccc}
S = 0.9 & | & 1 & | & 2 & | & 3 & | & \ldots \\
S = 0.5 & | & 0 & | & 1 & | & 2 & | & \ldots \\
S = 0.1 & | & 1 & | & 2 & | & 3 & | & \ldots \\
\end{array}
\]

On average, at \( n=3 \), the value is \( \frac{1}{3} \).

Variance Function

Def: The variance function of a random process \( X(t) \) is

\[ \text{Var}[X(t)] = \mathbb{E}\left[ (X(t) - m_X(t))^2 \right]. \]

Note: Both \( m_X(t) \) and \( \text{Var}[X(t)] \) are functions of \( t \).
**Auto correlation Function**

**Def.** The auto correlation function of \( X(t) \) is

\[
R_x(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)]
\]

Auto correlation function takes two time instants \( t_1 \) and \( t_2 \). Since \( X(t_1) \) and \( X(t_2) \) are two random variables, \( R_x(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)] \) measures the correlation of these two random variables.

**Corollary**

**Auto covariance Function**

**Def.** The auto covariance function of \( X(t) \) is

\[
C_x(t_1, t_2) = \mathbb{E}[(X(t_1) - m_x(t_1))(X(t_2) - m_x(t_2))]
\]

**Properties**

1. \( C_x(t_1, t_2) = R_x(t_1, t_2) - m_x(t_1)m_x(t_2) \)
2. \( C_x(t, t) = \text{Var}[X(t)] \).

**Proof:**

\[
C_x(t_1, t_2) = \mathbb{E}[(X(t_1) - m_x(t_1))(X(t_2) - m_x(t_2))]
\]

\[
= R_x(t_1, t_2) - m_x(t_1)m_x(t_1) - m_x(t_2)m_x(t_2) + m_x(t_1)m_x(t_2) + m_x(t_2)m_x(t_1)
\]

\[
= R_x(t_1, t_2) - m_x(t_1)m_x(t_1) - m_x(t_2)m_x(t_2) + m_x(t_1)m_x(t_2) + m_x(t_2)m_x(t_1)
\]

\[
= R_x(t_1, t_2) - m_x(t_1)m_x(t_1) - m_x(t_2)m_x(t_2) + m_x(t_1)m_x(t_2) + m_x(t_2)m_x(t_1)
\]

\[
C_x(t, t) = \mathbb{E}[(X(t) - m_x(t))^2] = \text{Var}[X(t)].
\]