Statistical Independence

**Def.** Two events $A$ and $B$ are statistically independent if $P(A \cap B) = P(A)P(B)$

**Interpretation through conditional probability**

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

If $A$ and $B$ are independent, then $P(A \cap B) = P(A)P(B)$.

Hence $P(A|B) = \frac{P(A)P(B)}{P(B)} = P(A)$.

So $A$ and $B$ are independent if the occurrence of $B$ provides no additional information about the occurrence of $A$.

**Remark:** We do not define independence as $P(A|B) = P(A)$.

1. $P(A \cap B) = P(A)P(B)$ holds even if $P(B) = 0$.
   - The conditional probability $P(A|B)$ is undefined when $P(B) = 0$.

2. Bayes theorem says $P(A|B)P(B) = P(B|A)P(A)$.
   - $P(A|B)$ is well-interpretable: if $B$ is the 1st experiment and $A$ is the second experiment, $A$ depends on $B$.
   - But what is $P(B|A)$? Does the 1st experiment depend on the 2nd experiment?
   - There is an inherent "cause-effect" difficulty.
**Example**

Roll a die twice

\[ A = \{ \text{1st die is 3} \} \quad B = \{ \text{2nd die is 4} \} \]

Is \( A \) and \( B \) independent?

Yes \( \because P(\text{get (3,4)}) = \frac{1}{36} \)

\[ P(A) = \frac{1}{6}, \quad P(B) = \frac{1}{6} \quad \therefore P(A \cap B) = P(A)P(B). \]

**Example**

\[ A = \{ \text{1st is 1} \} \quad B = \{ \text{Sum is 7} \} \]

Is \( A \) and \( B \) independent?

Yes \( \because P(A \cap B) = P\{(1,6)\} = \frac{1}{36} \)

\[ P(A) = \frac{1}{6}, \quad P(B) = P\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} = \frac{1}{6} \]

\[ \therefore P(A \cap B) = P(A)P(B). \]

**Example**

\[ A = \{ \text{max is 2} \} \quad B = \{ \text{min is 2} \} \]

No \( \because A = \{ (1,2), (2,1), (2,2) \} \)

\[ B = \{ (2,2), (2,3), (2,4), (2,5), (2,6), (3,2), (4,2), (5,2), (6,2) \} \]

\[ P(A) = \frac{3}{36}, \quad P(B) = \frac{9}{36}. \]

\[ P(A \cap B) = P\{(2,2)\} = \frac{1}{36}. \]
Example

Let \( \Omega = [0,1]^2 \). Consider three events

\[
\begin{align*}
A &= \{(x,y) \mid x > \frac{1}{2}\} \\
B &= \{(y) \mid y > \frac{1}{2}\} \\
C &= \{(x,y) \mid x > y\}.
\end{align*}
\]

(a) \( \mathbb{P}(A \cap B) = \mathbb{P}(\{(x,y) \mid x > \frac{1}{2}, y > \frac{1}{2}\}) = \frac{1}{4} \)
\( \mathbb{P}(A) = \frac{1}{2}, \quad \mathbb{P}(B) = \frac{1}{2}. \)
So \( \mathbb{P}(A \cap B) \neq \mathbb{P}(A) \mathbb{P}(B) \).

(b) \( \mathbb{P}(B \cap C) = \mathbb{P}(\{(x,y) \mid y > \frac{1}{2}, x < \frac{1}{2}\}) = \frac{1}{8} \)
\( \mathbb{P}(B) = \frac{1}{2}, \quad \mathbb{P}(C) = \frac{1}{2}, \) So \( \mathbb{P}(B \cap C) \neq \mathbb{P}(B) \mathbb{P}(C) \).

Remark

Exclusive VS Independence.

If \( A \) and \( B \) are exclusive, then \( A \cap B = \emptyset \).
So \( \mathbb{P}(A \cap B) = 0. \) This does not imply that
\( \mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B) \), unless \( \mathbb{P}(A) = 0 \) or \( \mathbb{P}(B) = 0 \).

If \( A \) and \( B \) are independent, then
\( \mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B) \). This does not imply \( \mathbb{P}(A \cap B) = 0 \), unless \( \mathbb{P}(A) = 0 \) or \( \mathbb{P}(B) = 0 \).
Permutation and Combination

Permutation and combination are two very basic ideas to help you count the number of possible outcomes. We will not go into details because many of you have learned it before.

Permutation

Suppose that there are $n$ objects with distinct labels. Pick $k$ from these $n$ objects. How many different ways can you find?

**Solution**

$$n \ (n-1) \ (n-2) \ \ldots \ \ (n-k+1)$$

1st choice 2nd choice

n options n-1 options

**Definition**

Define factorial "!" as

$$n! \overset{df}{=} n \ (n-1) \ (n-2) \ \ldots \ (2) \ (1),$$

and define $0! = 1$.

Therefore,

$$n \ (n-1) \ (n-2) \ \ldots \ (n-k+1)$$

$$= \frac{n!}{(n-k)!}$$

We denote this number as $P_k$. 

Example

4 balls \{1, 2, 3, 4\}. Choose 2.

All possible ways:

\[
\begin{array}{c}
1 & 2 \\
1 & 3 \\
1 & 4 \\
2 & 1 \\
2 & 3 \\
2 & 4 \\
3 & 1 \\
3 & 2 \\
3 & 4 \\
4 & 1 \\
4 & 2 \\
4 & 3
\end{array}
\]

12 possible options. Can be calculated as

\[4 \times 3\]

the 2nd ball
you only have 3 choices left.
the first ball
you have 4 choices.

Combination

Suppose that there are \(n\) different objects with distinct labels. Pick \(k\) from these \(n\) objects. Find the number of ways. Do not care about ordering.

Example

\(\{A, B, C\}\) \rightarrow get 2 from them.

All possible choices:

\[
\begin{array}{c}
AB \\
AC \\
BC \\
CA \\
BA
\end{array}
\]

However, \(AB\) and \(BA\) are the same because we do not care about the order.

\[
\begin{array}{c}
AC \\
BC \\
CA
\end{array}
\]

are the same because we do not care about the order.
Therefore, there are

\[ \frac{3!}{(3-2)!} = 6 \] permutations

but there are only

\[ \left( \frac{\frac{3!}{(3-2)!}}{2} \right) = 3 \] combinations.

In general, for \( \frac{n!}{(n-k)!} \) permutations, each permutation will have \( k! \) duplicates. Why \( k! \) duplicate? Consider

\[ \{ A, B, C \} \rightarrow \text{get 2} \]

\[
\begin{array}{c}
\text{ball 1} \quad \text{ball 2} \\
\text{ball 2} \quad \text{ball 1}
\end{array}
\]

\[ 2! \text{ in total.} \]

\[ \{ A, B, C, D \} \rightarrow \text{get 3} \]

\[
\begin{array}{c}
\text{ball 1} \quad \text{ball 2} \quad \text{ball 3} \\
\text{ball 2} \quad \text{ball 3} \\
\text{ball 2} \quad \text{ball 1}
\end{array}
\]

\[ 3! \text{ in total.} \]

Look at \( ABC \). How many ways can we rearrange \( ABC \)?

\[ ABC \quad BAC \quad CBA \\
ACB \quad BCA \quad CBA \]

\[ \uparrow \uparrow \uparrow \]

ball 1  ball 2  ball 3.