Basic Set Theory

In probability and statistics, we are interested in studying the likelihood of various "events", e.g. the probability we get a "Head" in a coin flip, or the probability we get a "5" in a die. In more complicated situations, we may even want to consider the co-occurrence of multiple events, e.g. "it rains" and "there is traffic", and "cell phone no battery". In order to study these situations, we need some basic ideas of sets.

• Set and Elements

So what is a set? A set is nothing but a collection of objects. Each object in a set is called an element. Mathematically, we use a capital letter to denote a set, and use a small greek letter to denote an element in a set:

\[ A = \{ s_1, s_2, \ldots, s_n \} \]

the set \( s \) elements contained in this set.

To write that \( s_3 \) is an element of the set \( A \), we use a short hand notation:

\[ s_3 \in A \quad \epsilon = "is \ an \ element \ of" \]
\textbf{Subset}

Very often we will not just study the entire set, but some portions of the set. For example, the set \( A \) could be all integers, and we might be interested in studying only the positive ones.

**Def**: \( B \) is a subset of \( A \) if every element in \( B \) is also an element in \( A \).

Mathematically, we write

\[ B \subseteq A \quad \text{if for every} \]

**Example**:

(i) \( A = \{1, 2, 3, 4, 5, 6\} \), \( B = \{1, 3, 5\} \)

(ii) \( A = \{\text{all real numbers}\} \), \( B = \{\text{all integers}\} \)

(iii) \( A = \{x : x \geq 0\} \) i.e. the positive interval.

\[ B = \{x : 1 < x < 3\}. \]

If \( B \) is a subset of \( A \) and \( B \) does not equal to \( A \), then we say

\[ B \subset A \quad \text{or} \quad \text{a proper subset} \]
Example:

(i) \( A = \{1, 2, 3, 4, 5, 6\} \)
\( B = \{1, 2, 3, 4, 5, 6\} \)
\( C = \{1, 4, 5\} \).

Then \( B \subseteq A \) (of course \( A \subseteq B \) and but \( C \subset A \). \( A = B \)).

(ii) \( A = \{x : x \geq 0\} \)
\( B = \{x : x > 0\} \)

Then \( B \subset A \) because 0 is not an element of B.

When will two set be equal?
\( A = B \) if and only if \( A \subseteq B \) and \( B \subseteq A \).

Union

The union of two sets \( A \) and \( B \) is the "OR" operation of two sets.

\[ A \cup B = \{x : x \in A \text{ or } x \in B\} \]
Pictorially, the union of the two sets is the total area occupied by either A or B or both:

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Example

(i) $A = \{1, 3, 4\}$, $B = \{2, 3, 5\}$,
then, $A \cup B = \{1, 2, 3, 4, 5\}$

(ii) $A = \{x : 3 < x \leq 4\}$
$B = \{x : x > 3.5\}$,
then $A \cup B = \{x : x > 3\}$

If $B \subseteq A$, then $A \cup B = A$ because $A$ is "bigger" than $B"
**Intersection**

The intersection of two sets $A$ and $B$ is the "AND" operation of two sets:

**Def**

$$A \cap B = \{ \xi : \xi \in A \text{ and } \xi \in B \}.$$  

**Example**

(i) $A = \{2, 5, 7, 8\}$  
$B = \{2, 7, 9, 10\}$  
$A \cap B = \{2, 7\}$

(ii) $A = \{x : -2 < x < 3\}$  
$B = \{x : 2 < x < 4\}$  
$A \cap B = \{x : 2 \leq x < 3\}$

(iii) $A = \{x : x \geq 0\}$  
$B = \{x : x \leq 0\}$  
$A \cap B = \{0\}$
- **Empty Set** $\emptyset$
  - A set that contains no element.

  (i) if $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, then $A \cap B = \emptyset$

  (ii) $A = \{2, 3, 4\}$, $B = \{5, 6\}$, $B = \emptyset$
    $A \cap B = \emptyset$
    $A \cup B = \{2, 3, 4\}$

  Note that $\emptyset$ is a subset of any set.

- **Complement**
  - Anything not in the set.

  Def $A^c = \{\mathcal{S} : \mathcal{S} \notin A\}$.

- **Universal Set**
  - A set that contains everything you are interested in.

  Notation: $\mathcal{S}$ or $\mathcal{U}$.
Example

(i) \( \Omega = \{ \text{all integers} \} \)
\[ A = \{ \text{even integers} \} \]
Then \( A^c = \{ \text{odd integers} \} \).

(ii) \( \Omega = \{ \text{all real numbers} \} \)
\[ A = \{ \text{rational numbers} \} \]
\[ A^c = \{ \text{irrational numbers} \} \]

(iii) \( \Omega = \{ x : -5 \leq x \leq 5 \} \)
\[ A = \{ x : x > 0 \} \quad 0.5 \leq x \leq 5 \]
\[ A^c = \{ x : -5 \leq x < 0 \} \]

\underline{Difference}
Something in \( A \) but not in \( B \).

Def \( A \setminus B = \{ x : x \in A \text{ and } x \notin B \} \)

Example:

(i) \( A = \{ 1, 2, 3, 4 \} \)
\[ B = \{ 2, 3, 7 \} \]
\[ A \setminus B = \{ 1, 4 \} \]

(ii) \( A = \{ x : x > 0 \} \) , \( B = \{ x : x < 2 \} \)
\[ A \setminus B = \{ x : 0 < x \leq 2 \} \]
**Disjoint**

**Def** Two sets $A$ and $B$ are disjoint if $A \cap B = \emptyset$.

If we have many sets $A_1, A_2, ..., A_n$, we say that \{\(A_1, \ldots, A_n\)\} is disjoint if $A_i \cap A_j = \emptyset$ for any pair of $i, j$ where $i \neq j$.

**Example:**

\[
\begin{align*}
A &= \{1, 2, 3\}, \\
B &= \{4, 5\}, \\
C &= \{6, 7\}, \\
\end{align*}
\]

then $A \cap B = \emptyset$, $A \cap C = \emptyset$, $B \cap C = \emptyset$.

So \{\(A, B, C\)\} is disjoint.

**Partition**

**Def** A collection of sets \{\(A_1, A_2, \ldots, A_n\)\} is a partition of a universal set $\Omega$ if

1. \{\(A_1, \ldots, A_n\)\} is disjoint
2. $A_1 \cup A_2 \cup \ldots \cup A_n = \Omega$.

Intuitively, (1) says every part in this collection is isolated by itself. There is no overlap. (2) says that by putting them together you will have the entire set.
Example

(i) \[ A_1 = \{ \text{all even integers} \} \quad \mathbb{Z} = \{ \text{all integers} \} \]
\[ A_2 = \{ \text{all odd integers} \} \]
Then \( \{ A_1, A_2 \} \) is a partition of \( \mathbb{Z} \).

(ii) \[ A_1 = \{ x : \ 0 \leq x \leq 5 \} \]
\[ A_2 = \{ x : \ 5 \leq x \leq 10 \} \quad \mathbb{Z} = \{ x : \ x \geq 0 \} \]
\[ A_3 = \{ x : \ x > 10 \} \]
\( \{ A_1, A_2, A_3 \} \) does not form a partition of \( \mathbb{Z} \) because \( A_1 \cap A_2 = \{ 5 \} \neq \emptyset \).

Properties of Set Operations

* **Commutative**: "order does not matter", or "symmetrical"
  \[ A \cup B = B \cup A \]
  \[ A \cap B = B \cap A \]

* **Associative**: "multiple union or intersect"
  \[ A \cup (B \cup C) = (A \cup B) \cup C \]
  \[ A \cap (B \cap C) = (A \cap B) \cap C \]

* **Distributive**: "mix of union & intersection"
  \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
  \[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
De Morgan's Law

\[(A \cap B)^c = A^c \cup B^c\]
\[(A \cup B)^c = A^c \cap B^c\]

Example

\[A = \{x : 0 \leq x \leq 1\}\]
\[B = \{x : 2 \leq x \leq 3\}\]
\[\Omega = \{x : x \geq 0\}\]

\[(A \cap B)^c = A^c \cup B^c\]
\[= \{x : x > 1\} \cup \{x : x < 2 \text{ or } x > 3\}\]
\[= \{x : x \geq 0\}\]

\[(A \cup B)^c = A^c \cap B^c\]
\[= \{x : 1 < x < 2\} \cap \{x : x > 3\}\]