Video 8

Fall 2017

Exercise 1.
Let

$$f_{X,Y}(x,y) = \begin{cases} ce^{-x}e^{-y}, & \text{if} \quad 0 \le y \le x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find c.

(b) Find $f_X(x)$ and $f_Y(y)$.

(c) Find $\mathbb{E}[X]$ and $\mathbb{E}[Y]$, $\text{Var}[X]$ and $\text{Var}[Y]$.

(d) Find $\mathbb{E}[XY]$, $\text{Cov}(X,Y)$ and ρ.

Exercise 2.
In class we have used the Cauchy-Schwarz inequality to show that $-1 \le \rho \le 1$. This exercise asks you to prove the Cauchy-Schwarz inequality:

$$(\mathbb{E}[XY])^2 \le \mathbb{E}[X^2]\mathbb{E}[Y^2].$$

Hint: Consider the expectation $\mathbb{E}[(tX+Y)^2]$. Note that it is a quadratic equation in t and $\mathbb{E}[(tX+Y)^2] \ge 0$ for all t. Consider the discriminant of this quadratic equation.

Exercise 3.
Let $\Theta \sim \text{Uniform}[0, 2\pi]$.

(a) If $X = \cos\Theta$, $Y = \sin\Theta$. Are X and Y uncorrelated?

(b) If $X = \cos(\Theta/4)$, $Y = \sin(\Theta/4)$. Are X and Y uncorrelated?

Exercise 4.
Let X and Y have a joint PDF

$$f_{X,Y}(x,y) = c(x+y),$$

for $0 \le x \le 1$ and $0 \le y \le 1$.

(a) Find c, $f_X(x)$, $f_Y(y)$, and $\mathbb{E}[Y]$.

(b) Find $f_{Y|X}(y|x)$.

(c) Find $\mathbb{P}[Y > X \mid X > 1/2]$.

(d) Find $\mathbb{E}[Y|X=x]$.

(e) Find $\mathbb{E}[\mathbb{E}[Y|X]]$, and compare with the $\mathbb{E}[Y]$ computed in (a).

Exercise 5.
Use Law of Total Expectation to compute the followings:

1. $\mathbb{E}[\sin(X+Y)]$, where $X \sim \mathcal{N}(0,1)$, and $Y \mid X \sim \text{Uniform}[x-\pi, x+\pi]$
2. $\mathbb{P}[Y < y]$, where $X \sim \text{Uniform}[0,1]$, and $Y \mid X \sim \text{Exponential}(x)$
3. $\mathbb{E}[Xe^Y]$, where $X \sim \text{Uniform}[-1,1]$, and $Y \mid X \sim \mathcal{N}(0,x^2)$