Video 10

Fall 2017

Exercise 1.
A Laplace random variable has a PDF

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \quad \text{where} \quad \lambda > 0,$$

and the variance is $\text{Var}[X] = \frac{2}{\lambda^2}$. Let $X_1, \ldots, X_{500}$ be a sequence of i.i.d. Laplace random variables. Let

$$M_{500} = \frac{X_1 + \ldots + X_{500}}{500}.$$

(a) Find $\mathbb{E}[X]$. Express your answer in terms of λ.

(b) Let $\lambda = 10$. Find, using Chebyshev inequality, a lower bound of

$$\mathbb{P}[-0.1 \leq M_{500} \leq 0.1].$$

(c) Let $\lambda = 10$. Find, using Central Limit Theorem, the probability

$$\mathbb{P}\left[-0.1 \leq M_{500} \leq 0.1\right].$$

You may leave your answer in terms of $\Phi(\cdot)$ function.

Exercise 2.
Let $X_1, \ldots, X_n$ be a sequence of iid random variables such that $X_i = \pm 1$ with equal probability. Let

$$Y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i.$$

Prove the Central Limit Theorem for this particular sequence of random variables by showing that

(a) $\mathbb{E}[Y_n] = 0$, $\text{Var}[Y_n] = 1$.

(b) The moment generating function of Y_n is $M_{Y_n}(s) \to e^{\frac{s^2}{2}}$ as $n \to \infty$.

Exercise 3.
Let $X_1, \ldots, X_N$ be a sequence of i.i.d. random variables with mean and variance

$$\mathbb{E}[X_n] = \mu, \quad \text{and} \quad \text{Var}[X_n] = \sigma^2, \quad n = 1, \ldots, N.$$

The distribution of X_n is, however, unknown. Let

$$M_N = \frac{1}{N} \sum_{n=1}^{N} X_n.$$

Use Central Limit Theorem to estimate the probability $\mathbb{P}[M_N > 2\mu]$

Exercise 4.
In class we derived Chebyshev inequality for the case of known variance σ^2. In this exercise we will prove a variant of the Chebyshev when the variance σ^2 is unknown but X is bounded between $a \le X \le b$.

(a) Let $\gamma \in \mathbb{R}$. Find a γ that minimizes $\mathbb{E}[(X-\gamma)^2]$. Hence, show that $\mathbb{E}[(X-\gamma)^2] \ge \text{Var}[X]$ for any γ.

(b) Let $\gamma = (a+b)/2$. Show that

$$\mathbb{E}[(X-\gamma)^2] = \mathbb{E}[(X-a)(X-b)] + \frac{(b-a)^2}{4}.$$

(c) From (a) and (b), show that $\text{Var}[X] \le \frac{(b-a)^2}{4}$.

(d) Show that for any $\varepsilon > 0$,

$$\mathbb{P}[|X-\mu| > \varepsilon] \le \frac{(b-a)^2}{4\varepsilon^2}.$$

Exercise 5.
Let $X_1, \ldots, X_N$ be a sequence of i.i.d. Bernoulli random variables with $\mathbb{P}[X_n = 1] = \theta$. Suppose that we have observed $x_1, \ldots, x_N$.

(a) Show that the PMF of X_n is $p_{X_n}(x_n \mid \theta) = \theta^{x_n}(1-\theta)^{1-x_n}$. Hence, find the joint PMF

$$p_{X_1,\ldots,X_N}(x_1, \ldots, x_N).$$

(b) Find the maximum likelihood estimate $\widehat{\theta}$, i.e.,

$$\widehat{\theta}_{\text{ML}} = \underset{\theta}{\text{argmax}} \;\; \log \, p_{X_1,\ldots,X_N}(x_1, \ldots, x_N).$$

Express your answer in terms of $x_1, \ldots, x_N$.

Exercise 6.
Let $Y_n = \theta + W_n$ be the output of a noisy channel where the input is a scalar θ and $W_n \sim \mathcal{N}(0,1)$ is an i.i.d. Gaussian noise. Suppose that we have observed $y_1, \ldots, y_N$.

(a) Express the PDF of Y_n in terms of θ and y_n. Hence, find the joint PDF of $Y_1, \ldots, Y_N$.

(b) Find the maximum likelihood estimate $\widehat{\theta}_{\text{ML}}$. Express your answer in terms of $y_1, \ldots, y_N$.