ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Spring 2017
Instructor: Prof. Stanley H. Chan

Mid Term 2

Spring 2017

Name: ______________________________ PUID: ______________

Please copy and write the following statement:

I certify that I have neither given nor received unauthorized aid on this exam.

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(Please copy and write the above statement.)

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(Signature)

Problem 1. (15 POINTS)
Determine whether the following statements are TRUE or FALSE. (A statement is true if it is always true. Otherwise we will say that the statement is false.) Circle your answer. No partial credit will be given.

1. If $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$, then X and Y are independent.

   TRUE or FALSE.

2. If X and Y are uncorrelated, then the following equation is valid.

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y].$$

   TRUE or FALSE.

3. A CDF F_X is right continuous at $x = b$ means that

$$\lim_{h \to 0} F_X(b + h) = F_X(b).$$

TRUE or FALSE.

4. Let X and Y be a pair of random variables. Then,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \left(\frac{d}{dx} F_{X,Y}(x, \infty) \right) dx.$$

TRUE or FALSE.

5. Let X be a random variable with CDF $F_X(x)$. Let $Y = 2X + 3$. Then,

$$F_Y(y) = F_X\left(\frac{y+3}{2}\right).$$

TRUE or FALSE.

Problem 2. (25 POINTS)
Multiple Choice. Please **circle** your answer.

1. Let X be a Gaussian random variable with mean $\mu = 5$ and variance $\sigma^2 = 9$. Denote $\Phi(\cdot)$ as the standard Gaussian CDF. Then, the probability $\mathbb{P}[4 \leq X \leq 8]$ is

   (a) $\Phi(1) - \Phi\left(\frac{-1}{3}\right)$
   (b) $\Phi\left(\frac{1}{3}\right) - \Phi\left(\frac{-1}{9}\right)$
   (c) $\Phi\left(\frac{8}{3}\right) - \Phi\left(\frac{4}{3}\right)$
   (d) $\Phi\left(\frac{-1}{3}\right) + \Phi(1)$
   (e) $\Phi\left(\frac{1}{\sqrt{3}}\right) - \Phi\left(-\frac{1}{\sqrt{3}}\right)$
   (f) $\Phi(1) - \Phi(-1)$
   (g) $\Phi(8) - \Phi(4)$
   (h) $\Phi\left(\frac{8}{3} - 5\right) - \Phi\left(\frac{4}{3} - 5\right)$
   (i) $\Phi\left(\frac{8}{9} - 5\right) - \Phi\left(\frac{4}{9} - 5\right)$
   (j) None of the above

2. Let X be a random variable with CDF

$$F_X(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{2}, & -1 \leq x < 0 \\ 1 - \frac{1}{4}e^{-2x}, & x \geq 0. \end{cases}$$

Then, $\mathbb{P}[X = 0]$ is

(a) 1
(b) $\frac{1}{2}$
(c) $\frac{1}{3}$
(d) $\frac{1}{4}$
(e) $\frac{1}{8}$
(f) $\frac{2}{3}$
(g) $\frac{3}{2}$
(h) $\frac{3}{4}$
(i) Problem undefined
(j) None of the above

3. Let X and Y be a joint Gaussian with PDF

$$f_{X,Y}(x,y) \propto \exp\left\{-\frac{1}{2}\left(4x^2 + y^2 + 2xy\right)\right\}$$

Then, $\mathrm{Var}[Y]$ is

(a) 1
(b) 1/2
(c) −1/2
(d) 1/3
(e) −1/3
(f) 3/4
(g) −3/4
(h) 4/3
(i) −4/3
(j) None of the above

4. Let X and Y be two independent random variables

$$X \sim \text{Bernoulli}(p), \qquad Y \sim \text{Poisson}(\lambda)$$

Let $Z = 3XY$. Find the moment generating function $M_Z(s) \stackrel{\text{def}}{=} \mathbb{E}[e^{sZ}]$.

(a) $3(1-p+pe^s)(e^{\lambda(e^s-1)})$
(b) $(1-p+pe^{3s})(e^{\lambda(e^{3s}-1)})$
(c) $3pe^{\lambda(e^s-1)} + (1-p)$
(d) $pe^{\lambda(e^s-1)} + (1-p)$
(e) $pe^{\lambda(e^{3s}-1)} + (1-p)$
(f) $pe^{\lambda(e^s-1)} + (1-p)e^{\lambda(e^{3s}-1)}$
(g) $3(1-p)e^{\lambda(e^s-1)}$
(h) $1-p+pe^s+3e^{\lambda(e^s-1)}$
(i) $3(1-p+pe^s+e^{\lambda(e^s-1)})$
(j) None of the above

5. Let X and Y be two random variables with PDFs

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad f_Y(y) = \frac{1}{\pi(1+y^2)}$$

Let $Z = X + Y$. The characteristic function $\Phi_Z(j;w) \stackrel{\text{def}}{=} \mathbb{E}[e^{-jwZ}]$ is

(a) $e^{-|w|}$
(b) $\pi e^{-|w|}$
(c) $\pi^2 e^{-|w|}$
(d) $e^{-2|w|}$
(e) $\pi e^{-2|w|}$
(f) $\pi^2 e^{-2|w|}$
(g) $2e^{-|w|}$
(h) $2\pi e^{-|w|}$
(i) $2\pi^2 e^{-|w|}$
(j) None of the above

Problem 3. (20 POINTS)
Let X and Y be a pair of random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} e^{-x}e^{-y}, & \text{if} \quad 0 \le x < \infty, \ \ 0 \le y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(a) (6 points). Find $f_X(x)$ and $f_Y(y)$. Are X and Y independent?

$f_X(x) =$	$f_Y(y) =$
Are X and Y independent? (circle one) YES / NO	

(b) (4 points). Find $\mathbb{E}[X]$ and $\mathbb{E}[Y]$, $\text{Var}[X]$ and $\text{Var}[Y]$.

$\mathbb{E}[X] =$	$\mathbb{E}[Y] =$
$\text{Var}[X] =$	$\text{Var}[Y] =$

(c) (10 points). Find $\text{Cov}(X-1, Y+1)$.

$\text{Cov}(X-1, Y+1) =$

Problem 4. (20 POINTS)
Let X and Y be two independent random variables, and let

$$f_X(x) = \begin{cases} xe^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad f_Y(y) = \begin{cases} ye^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases}. \tag{1}$$

Let $Z = X + Y$.

(a) (7 points) It is known that the CDF $F_Z(z)$ can be expressed in the following integration. Find the integration limit a.

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{a} f_X(x) f_Y(y) dx dy$$

$a =$

(b) (7 points) It is known that the PDF $f_Z(z)$ can be expressed in the following integration. Find the integration limits b and c for the specific $f_X(x)$ and $f_Y(y)$ given in (1). (Hint: If you write $b = -\infty$ and $c = \infty$, then you are wrong. You need to take care of the range of y.)

$$f_Z(z) = \int_{b}^{c} f_X(z - y) f_Y(y) dy$$

$b =$ $c =$

(c) (6 points) Using the results in (a) and (b), find the PDF $f_Z(z)$.

$$f_Z(z) = \begin{cases} \qquad\qquad\qquad\qquad , & z \geq 0, \\ \qquad\qquad\qquad\qquad , & z < 0. \end{cases}$$

Problem 5. (20 POINTS)
Consider three random variables X, Y and Z with the following conditional distributions.

- $X \mid Y \sim \text{Exponential}\left(\frac{1}{Y}\right)$. That is, the PDF of X given $Y = y$ is $f_{X|Y}(x|y) = \frac{1}{y}e^{-\frac{x}{y}}$.
- $Y \mid Z \sim \mathcal{N}(Z, 1)$. That is, the PDF of Y given $Z = z$ is $f_{Y|Z}(y|z) = \frac{1}{\sqrt{2\pi}}e^{-(y-z)^2/2}$.
- $Z \sim \text{Bernoulli}(p)$. That is, the PMF of Z is $p_Z(1) = p$ and $p_Z(0) = 1 - p$.

(a) (10 points). Find $\mathbb{E}[Y]$.

$\mathbb{E}[Y] =$

(b) (10 points). Find $\mathbb{E}[X]$.

$\mathbb{E}[X] =$

Useful Identities

1. $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \ldots = \frac{1}{1-r}$
2. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
3. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$
4. $\sum_{k=1}^{\infty} kr^{k-1} = 1 + 2r + 3r^2 + \ldots = \frac{1}{(1-r)^2}$
5. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^3 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$
6. $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Common Distributions

Bernoulli	$\mathbb{P}[X=1] = p$	$\mathbb{E}[X] = p$	$\text{Var}[X] = p(1-p)$	$M_X(s) = 1 - p + pe^s$
Binomial	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$	$\mathbb{E}[X] = np$	$\text{Var}[X] = np(1-p)$	$M_X(s) = (1 - p + pe^s)^n$
Geometric	$p_X(k) = p(1-p)^{k-1}$	$\mathbb{E}[X] = \frac{1}{p}$	$\text{Var}[X] = \frac{1-p}{p^2}$	$M_X(s) = \frac{pe^s}{1-(1-p)e^s}$
Poisson	$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$	$\mathbb{E}[X] = \lambda$	$\text{Var}[X] = \lambda$	$M_X(s) = e^{\lambda(e^s - 1)}$
Gaussian	$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mathbb{E}[X] = \mu$	$\text{Var}[X] = \sigma^2$	$M_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$
Exponential	$f_X(x) = \lambda \exp\{-\lambda x\}$	$\mathbb{E}[X] = \frac{1}{\lambda}$	$\text{Var}[X] = \frac{1}{\lambda^2}$	$M_X(s) = \frac{\lambda}{\lambda - s}$
Uniform	$f_X(x) = \frac{1}{b-a}$	$\mathbb{E}[X] = \frac{a+b}{2}$	$\text{Var}[X] = \frac{(b-a)^2}{12}$	$M_X(s) = \frac{e^{sb} - e^{sa}}{s(b-a)}$

Fourier Transform Table

	$f(t) \longleftrightarrow F(w)$		$f(t) \longleftrightarrow F(w)$
1.	$e^{-at}u(t) \longleftrightarrow \frac{1}{a+jw}$, $a > 0$	10.	$\text{sinc}^2(\frac{Wt}{2}) \longleftrightarrow \frac{2\pi}{W}\Delta(\frac{w}{2W})$
2.	$e^{at}u(-t) \longleftrightarrow \frac{1}{a-jw}$, $a > 0$	11.	$e^{-at}\sin(w_0 t)u(t) \longleftrightarrow \frac{w_0}{(a+jw)^2 + w_0^2}$, $a > 0$
3.	$e^{-a\|t\|} \longleftrightarrow \frac{2a}{a^2 + w^2}$, $a > 0$	12.	$e^{-at}\cos(w_0 t)u(t) \longleftrightarrow \frac{a+jw}{(a+jw)^2 + w_0^2}$, $a > 0$
4.	$\frac{a^2}{a^2 + t^2} \longleftrightarrow \pi a e^{-a\|w\|}$, $a > 0$	13.	$e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sqrt{2\pi}\sigma e^{-\frac{\sigma^2 w^2}{2}}$
5.	$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+jw)^2}$, $a > 0$	14.	$\delta(t) \longleftrightarrow 1$
6.	$t^n e^{-at}u(t) \longleftrightarrow \frac{n!}{(a+jw)^{n+1}}$, $a > 0$	15.	$1 \longleftrightarrow 2\pi\delta(w)$
7.	$\text{rect}(\frac{t}{\tau}) \longleftrightarrow \tau\text{sinc}(\frac{w\tau}{2})$	16.	$\delta(t - t_0) \longleftrightarrow e^{-jwt_0}$
8.	$\text{sinc}(Wt) \longleftrightarrow \frac{\pi}{W}\text{rect}(\frac{w}{2W})$	17.	$e^{jw_0 t} \longleftrightarrow 2\pi\delta(w - w_0)$
9.	$\Delta(\frac{t}{\tau}) \longleftrightarrow \frac{\tau}{2}\text{sinc}^2(\frac{w\tau}{4})$		

Some definitions:

$$\text{sinc}(t) = \frac{\sin(t)}{t} \qquad \text{rect}(t) = \begin{cases} 1, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases} \qquad \Delta(t) = \begin{cases} 1 - 2|t|, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

Basic Trigonometry

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = 2\cos^2\theta - 1.$$

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \quad \sin A \sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$$
$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B)) \quad \cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$