Mid Term 2

Fall 2017

Name: ______________________________ PUID: ________________

Please copy and write the following statement:

I certify that I have neither given nor received unauthorized aid on this exam.

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(Please copy and write the above statement.)

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(Signature)

Problem 1. (25 POINTS)
Determine whether the following statements are TRUE or FALSE. (A statement is true if it is always true. Otherwise we will say that the statement is false.) Circle your answer. No partial credit will be given.

1. The statement "CDF is right-continuous at $x = b$" means that

$$\lim_{h \to 0} F_X(b+h) = F_X(b).$$

TRUE or FALSE.

2. Weak Law of Large Number states that the sample mean M_N of a sequence of i.i.d. random variables $X_1, \ldots, X_N$:

$$M_N = \frac{1}{N}\sum_{n=1}^{N} X_n$$

will converge in distribution to a standard Gaussian as $N \to \infty$.

TRUE or FALSE.

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3. Let X and Y be two independent random variables with PDF $f_X(x)$ and $f_Y(y)$. Then the PDF of $Z = X + Y$ is

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy.$$

TRUE or FALSE.

4. If $\mathbb{E}[XY] = 0$, then we must have $\mathbb{E}[X] = 0$, or $\mathbb{E}[Y] = 0$, or $\mathbb{E}[X] = \mathbb{E}[Y] = 0$.

TRUE or FALSE.

5. The moment generating function exists for any random variable.

TRUE or FALSE.

Problem 2. (25 POINTS)
Multiple Choice. Please **circle** your answer.

1. Let $X \sim \mathcal{N}(\mu, \sigma^2)$ where $\mu = 3$ and $\sigma = 4$. Then $\mathbb{P}[X \geq 2 \cup X \leq 1]$ is
   (a) $\Phi\left(-\frac{1}{4}\right) - \Phi\left(-\frac{1}{2}\right)$
   (b) $\Phi\left(-\frac{1}{4}\right) + \Phi\left(-\frac{1}{2}\right)$
   (c) $1 - \Phi\left(-\frac{1}{16}\right) + \Phi\left(-\frac{1}{4}\right)$
   (d) $1 - \Phi\left(-\frac{1}{4}\right) + \Phi\left(-\frac{1}{2}\right)$
   (e) $\Phi\left(-\frac{1}{16}\right) - \Phi\left(-\frac{1}{4}\right)$
   (f) $\Phi\left(-\frac{1}{16}\right) + \Phi\left(-\frac{1}{4}\right)$
   (g) $1 - \Phi\left(-\frac{1}{2}\right) + \Phi\left(-\frac{1}{4}\right)$
   (h) $1 - \Phi\left(-\frac{1}{4}\right) + \Phi\left(-\frac{1}{16}\right)$
   (i) Problem undefined
   (j) None of the above

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2. Let X and Y be joint Gaussian with PDF

$$f_{X,Y}(x,y) \propto \exp\left\{-\frac{1}{2}\left(3x^2 - 2y^2 + 2xy\right)\right\}$$

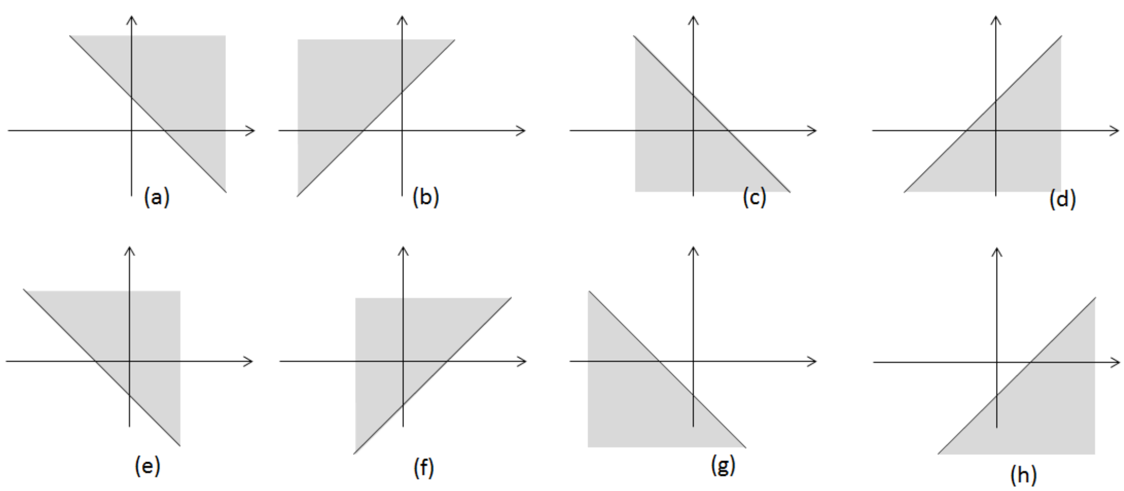
Then $\text{Var}[X]$, $\text{Var}[Y]$ and $\text{Cov}(X,Y)$ are

(a) $\text{Var}[X] = 3/7$, $\text{Var}[Y] = -2/7$, $\text{Cov}(X,Y) = 1/7$
(b) $\text{Var}[X] = 3/7$, $\text{Var}[Y] = 2/7$, $\text{Cov}(X,Y) = 1/7$
(c) $\text{Var}[X] = 2/7$, $\text{Var}[Y] = -3/7$, $\text{Cov}(X,Y) = 1/7$
(d) $\text{Var}[X] = 2/7$, $\text{Var}[Y] = 3/7$, $\text{Cov}(X,Y) = 1/7$
(e) $\text{Var}[X] = 3/7$, $\text{Var}[Y] = 2/7$, $\text{Cov}(X,Y) = -1/7$
(f) $\text{Var}[X] = 3/7$, $\text{Var}[Y] = -2/7$, $\text{Cov}(X,Y) = -1/7$
(g) $\text{Var}[X] = 2/7$, $\text{Var}[Y] = 3/7$, $\text{Cov}(X,Y) = -1/7$
(h) $\text{Var}[X] = 2/7$, $\text{Var}[Y] = -3/7$, $\text{Cov}(X,Y) = 1/7$
(i) Problem undefined
(j) None of the above

Hint: The 2×2 inverse of a matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$$

3. Let X and Y be two independent random variables, and let $Z = X - Y$. Assume $Z > 0$. Which one of the following shaded regions should we evaluate the CDF $F_Z(z) = \mathbb{P}[Z \leq z]$?



Remark: There is no option (i) and (j) for this problem.

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4. Let $X_i \sim \text{Exponential}(\lambda)$ be i.i.d. random variables for $i = 1, \ldots, n$. Let $Y = X_1 + \ldots + X_n$. Find $f_Y(y)$. Hint: Use characteristic function.

   (a) $n\lambda e^{-\lambda y}$, for $y \geq 0$
   (b) $(\lambda e^{-\lambda y})^n$, for $y \geq 0$
   (c) $y^n e^{-y}$, for $y \geq 0$
   (d) ye^{-y}, for $y \geq 0$
   (e) $y^n e^{-ny}$, for $y \geq 0$
   (f) $\frac{\lambda^n}{n!} y^n e^{-\lambda y}$, for $y \geq 0$
   (g) $\frac{\lambda^n}{(n-1)!} y^{n-1} e^{-\lambda y}$, for $y \geq 0$
   (h) $\lambda^n y^{n-1} e^{-\lambda y}$, for $y \geq 0$
   (i) Problem undefined
   (j) None of the above

5. Let $X_i \sim \text{Binomial}(n, p)$ for $i = 1, \ldots, N$. Let

$$Y = \frac{1}{N} \sum_{i=1}^{N} X_i.$$

   By Chebyshev inequality, what is the minimum N that will make

$$\mathbb{P}[|Y - \mathbb{E}[Y]| > 0.1] \leq 0.1,$$

   where $n = 10$, $p = 1/2$.

   (a) $N \geq 100$
   (b) $N \geq 200$
   (c) $N \geq 250$
   (d) $N \geq 1000$
   (e) $N \geq 2000$
   (f) $N \geq 2500$
   (g) $N \geq 5000$
   (h) $N \geq 25000$
   (i) Problem undefined
   (j) None of the above

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Problem 3. (15 POINTS)
Let $X \sim \mathcal{N}(0,1)$, and let $Y \mid X \sim \mathcal{N}(X,1)$.
Find $\mathbb{P}[Y > 1]$. Express your answer in terms of the standard Gaussian $\Phi(\cdot)$ function.

Hint: $(x-y)^2 + x^2 = 2\left[\left(x - \frac{y}{2}\right)^2 + \frac{y^2}{4}\right]$

$\mathbb{P}[Y > 1] =$

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Problem 4. (20 POINTS)
Throw a dice twice. Let I_1 be the first number, and I_2 be the second number. Define

$$X = \max(I_1, I_2), \qquad Y = \min(I_1, I_2). \tag{1}$$

(a) (8 points) Find the joint PMF of X and Y.

Answer: Please write down $p_{X,Y}(x,y)$ in the following table.

$X = \max(I_1, I_2)$

$Y = \min(I_1, I_2)$

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

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(b) (4 points) Is X and Y independent? (No point without explanation)

YES / NO

(c) (4 points) Find $\mathbb{P}[X \leq 5 \mid Y \geq 4]$.

$\mathbb{P}[X \leq 5 \mid Y \geq 4] =$

(d) (4 points) Find $\mathbb{E}[Y \mid X = 3]$.

$\mathbb{E}[Y \mid X = 3] =$

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Problem 5. (15 POINTS)
Let X be a random variable with PDF

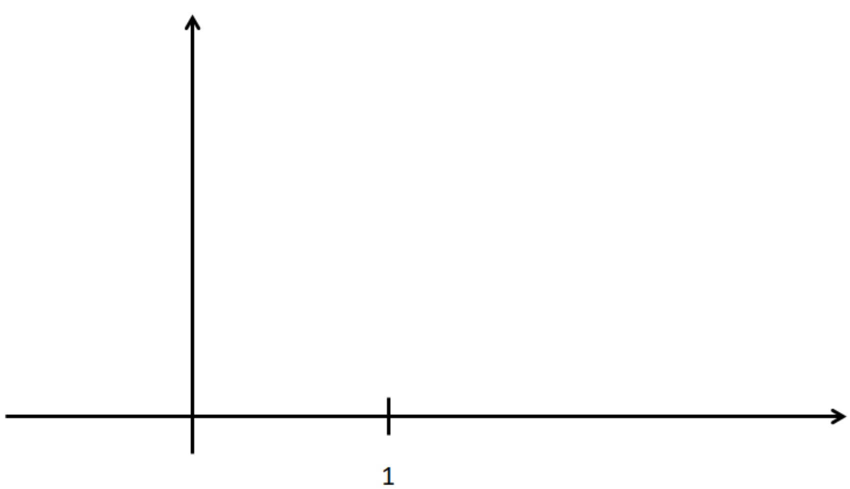
$$f_X(x) = e^{-x}, \qquad x \geq 0.$$

Define

$$Y = \begin{cases} X, & X \leq 1 \\ 1, & X > 1. \end{cases}$$

(a) (5 points) Draw $f_Y(y)$.

Answer: Please draw $f_Y(y)$ in the following figure. Mark the height of delta functions, if exist.



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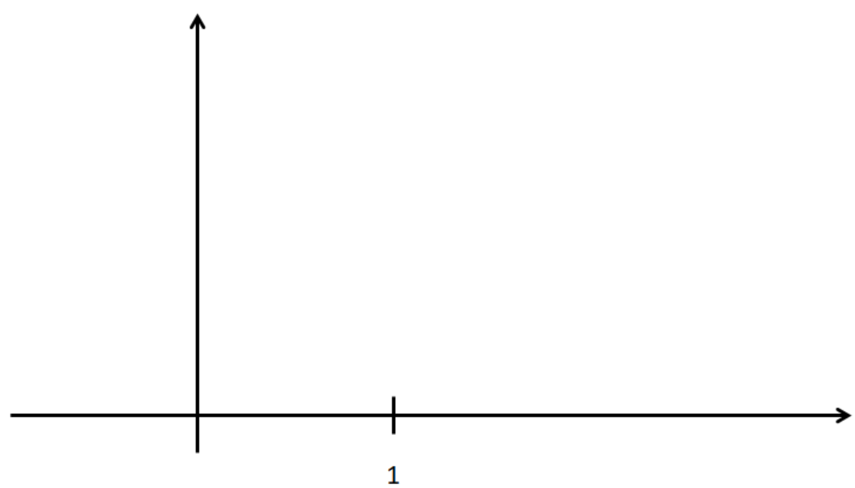
(b) (5 points) Draw $F_Y(y)$.

Answer: Please draw $F_Y(y)$ in the following figure. Mark the discontinuous points clearly with empty dots and solid dots. Mark the height of the jumps.



(c) (5 points) Find the moment generating function of Y.

$M_Y(s) =$

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Useful Identities

1. $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \ldots = \frac{1}{1-r}$
2. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
3. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$
4. $\sum_{k=1}^{\infty} kr^{k-1} = 1 + 2r + 3r^2 + \ldots = \frac{1}{(1-r)^2}$
5. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^3 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$
6. $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Common Distributions

Bernoulli	$\mathbb{P}[X=1] = p$	$\mathbb{E}[X] = p$	$\text{Var}[X] = p(1-p)$	$M_X(s) = 1 - p + pe^s$
Binomial	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$	$\mathbb{E}[X] = np$	$\text{Var}[X] = np(1-p)$	$M_X(s) = (1 - p + pe^s)^n$
Geometric	$p_X(k) = p(1-p)^{k-1}$	$\mathbb{E}[X] = \frac{1}{p}$	$\text{Var}[X] = \frac{1-p}{p^2}$	$M_X(s) = \frac{pe^s}{1-(1-p)e^s}$
Poisson	$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$	$\mathbb{E}[X] = \lambda$	$\text{Var}[X] = \lambda$	$M_X(s) = e^{\lambda(e^s - 1)}$
Gaussian	$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mathbb{E}[X] = \mu$	$\text{Var}[X] = \sigma^2$	$M_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$
Exponential	$f_X(x) = \lambda \exp\{-\lambda x\}$	$\mathbb{E}[X] = \frac{1}{\lambda}$	$\text{Var}[X] = \frac{1}{\lambda^2}$	$M_X(s) = \frac{\lambda}{\lambda - s}$
Uniform	$f_X(x) = \frac{1}{b-a}$	$\mathbb{E}[X] = \frac{a+b}{2}$	$\text{Var}[X] = \frac{(b-a)^2}{12}$	$M_X(s) = \frac{e^{sb} - e^{sa}}{s(b-a)}$

Fourier Transform Table

	$f(t) \longleftrightarrow F(w)$		$f(t) \longleftrightarrow F(w)$
1.	$e^{-at}u(t) \longleftrightarrow \frac{1}{a+jw}$, $a > 0$	10.	$\text{sinc}^2(\frac{Wt}{2}) \longleftrightarrow \frac{2\pi}{W}\Delta(\frac{w}{2W})$
2.	$e^{at}u(-t) \longleftrightarrow \frac{1}{a-jw}$, $a > 0$	11.	$e^{-at}\sin(w_0 t)u(t) \longleftrightarrow \frac{w_0}{(a+jw)^2 + w_0^2}$, $a > 0$
3.	$e^{-a\lvert t\rvert} \longleftrightarrow \frac{2a}{a^2+w^2}$, $a > 0$	12.	$e^{-at}\cos(w_0 t)u(t) \longleftrightarrow \frac{a+jw}{(a+jw)^2 + w_0^2}$, $a > 0$
4.	$\frac{a^2}{a^2+t^2} \longleftrightarrow \pi a e^{-a\lvert w\rvert}$, $a > 0$	13.	$e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sqrt{2\pi}\sigma e^{-\frac{\sigma^2 w^2}{2}}$
5.	$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+jw)^2}$, $a > 0$	14.	$\delta(t) \longleftrightarrow 1$
6.	$t^n e^{-at}u(t) \longleftrightarrow \frac{n!}{(a+jw)^{n+1}}$, $a > 0$	15.	$1 \longleftrightarrow 2\pi\delta(w)$
7.	$\text{rect}(\frac{t}{\tau}) \longleftrightarrow \tau\text{sinc}(\frac{w\tau}{2})$	16.	$\delta(t - t_0) \longleftrightarrow e^{-jwt_0}$
8.	$\text{sinc}(Wt) \longleftrightarrow \frac{\pi}{W}\text{rect}(\frac{w}{2W})$	17.	$e^{jw_0 t} \longleftrightarrow 2\pi\delta(w - w_0)$
9.	$\Delta(\frac{t}{\tau}) \longleftrightarrow \frac{\tau}{2}\text{sinc}^2(\frac{w\tau}{4})$		

Some definitions:

$$\text{sinc}(t) = \frac{\sin(t)}{t} \qquad \text{rect}(t) = \begin{cases} 1, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases} \qquad \Delta(t) = \begin{cases} 1 - 2|t|, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

Basic Trigonometry

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = 2\cos^2\theta - 1.$$

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \qquad \sin A \sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$$

$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B)) \qquad \cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$

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