Mid Term 1

Spring 2017

Name: ______________________________ PUID: ______________

Please copy and write the following statement:

I certify that I have neither given nor received unauthorized aid on this exam.

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(Please copy and write the above statement.)

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(Signature)

Problem 1. (15 POINTS)
Determine whether the following statements are TRUE or FALSE. (A statement is true if it is always true. Otherwise we will say that the statement is false.) Circle your answer. No partial credit will be given.

1. For any sets A and B, it holds that $\mathbb{P}[A \cap B] = \mathbb{P}[A \cup B] - \mathbb{P}[A \backslash B] - \mathbb{P}[B \backslash A]$.

   TRUE or FALSE.

2. For any $0 < p < 1$,
$$\sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np.$$

   TRUE or FALSE.

3. Let A and B be two events. If A and B are independent and disjoint, then $A = \emptyset$ or $B = \emptyset$ or $A = B = \emptyset$.

   TRUE or FALSE.

4. Let $X \sim \text{Poisson}(\lambda)$. Then $\mathbb{E}[(2X-1)^2] = 4\lambda^2 + 1$.

TRUE or FALSE.

5. Throw a dice twice. Let X be the outcome of the first dice, and Y be the outcome of the second dice. The probability $\mathbb{P}[\max(X, Y) = 2]$ is $1/12$.

TRUE or FALSE.

Problem 2. (25 POINTS)
Multiple Choice. Please **circle** your answer.

1. Let X be a random variable with PMF

$$p_X(-1) = \frac{1}{3}, \;\; p_X(0) = \frac{1}{4}, \;\; p_X(1) = \frac{1}{4}, \;\; p_X(2) = \frac{1}{6}.$$

Find $\mathbb{E}[4X^2]$.

(a) 0
(b) 1
(c) 4
(d) 5
(e) 1/36
(f) 1/24
(g) 1/72
(h) 2/27
(i) 7/24
(j) 7/6
(k) None of the above

2. Let $p_X(k) = c/2^k$ for $k = 0, 2, 4, 6, \ldots$. Then, $c =$

(a) 1
(b) 2
(c) 4
(d) $\frac{1}{2}$
(e) $\frac{1}{4}$
(f) $\frac{1}{8}$
(g) $\frac{2}{3}$
(h) $\frac{3}{2}$
(i) $\frac{3}{4}$
(j) $\frac{4}{3}$

3. A company makes three types of pens: Type A, Type B and Type C. Among all the pens, 40% are Type A, 30% are Type B, and 30% are Type C. The probability that Type A has defect is 0.2, that Type B has defect is 0.1, and that Type C has defect is 0.1. Suppose that you get a pen and see that it is defective. Find the probability that the pen was Type A.

   (a) 0.4
   (b) 0.2
   (c) 0.1
   (d) 0.16
   (e) 0.14
   (f) 0.12
   (g) 0.08
   (h) 0.04
   (i) 0.02
   (j) None of the above

4. Let A, B, C be events with probabilities $\mathbb{P}[A] = 0.3$, $\mathbb{P}[B] = 0.2$, $\mathbb{P}[C] = 0.5$. Find $\mathbb{P}[A \cup B \cup C]$ if A, B and C are independent

   (a) 0
   (b) 0.06
   (c) 0.22
   (d) 0.36
   (e) 0.44
   (f) 0.5
   (g) 0.72
   (h) 0.8
   (i) 0.94
   (j) None of the above

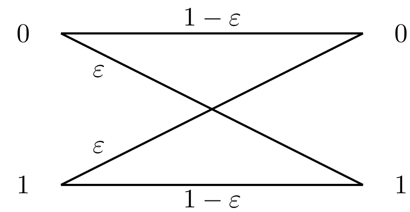
5. Let $X \sim \text{Poisson}(\lambda)$, and let

$$Y = \begin{cases} 1, & \text{if } X \geq 1, \\ 0, & \text{if } X < 1. \end{cases}$$

   Find $\mathbb{E}[Y]$.

   (a) λ
   (b) $1 + \lambda$
   (c) $1 - \lambda$
   (d) $\frac{1}{\lambda}$
   (e) $1 - \frac{1}{\lambda}$
   (f) $1 - \lambda^2$
   (g) $e^{-\lambda}$
   (h) $1 - e^{-\lambda}$
   (i) $1 - 2e^{-\lambda}$
   (j) None of the above

Problem 3. (30 POINTS)
Consider the following communication channel. A source transmits a string of binary symbols through a noisy communication channel. Each symbol is 0 or 1 with probability p and $1-p$, respectively, and is received with an error probability ε. Assume that error in different symbols transmissions is independent.



There are two transmission schemes:

- Basic Scheme: When you want to transmit "0", you send "0" through the channel. At the receiver side, if you obtain a "0" then you claim you have received a "0".
- Advanced Scheme: When you want to transmit "0", you send it as "000". At the receiver side, you make a majority vote. That is, if there are at least two 0's, you claim you have received a "0". Similarly, if you want to transmit "1", you send it as "111". After the majority vote on the receiver side, you claim that you have received a "1" if there are at least two 1's.

(a) (8 points) What is the probability that a symbol is correctly received using Basic Scheme? (Hint: Law of Total Probability)

(b) (14 points) What is the probability that a symbol is correctly received using the Advanced Scheme? (Hint: Law of Total Probability) continue ...

(c) (8 points) Find the maximum ε that the Advanced Scheme performs better than the Basic Scheme.

Problem 4. (30 POINTS)
Two dice are tossed. Let X be the absolute difference in the number of dots facing up. For example, if the first dice is 4 and the second dice is 1, the absolute difference is $|4-1| = 3$.

(a) (10 points) Find the PMF of X.

(b) (6 points) Find the probability $\mathbb{P}[X \leq 2]$.

(c) (8 points) Find the conditional probability of $X \leq 2$ given that the 1st dice is 2. That is, find

$$\mathbb{P}[X \leq 2 \mid \text{1st dice is 2}].$$

(d) (6 points) Find $\mathbb{E}[X]$.

Useful Identities

1. $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \ldots = \frac{1}{1-r}$
2. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
3. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$
4. $\sum_{k=1}^{\infty} kr^{k-1} = 1 + 2r + 3r^2 + \ldots = \frac{1}{(1-r)^2}$
5. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^3 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$
6. $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Common Distributions

Bernoulli	$\mathbb{P}[X=1] = p$	$\mathbb{E}[X] = p$	$\text{Var}[X] = p(1-p)$	$M_X(s) = 1 - p + pe^s$
Binomial	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$	$\mathbb{E}[X] = np$	$\text{Var}[X] = np(1-p)$	$M_X(s) = (1-p+pe^s)^n$
Geometric	$p_X(k) = p(1-p)^{k-1}$	$\mathbb{E}[X] = \frac{1}{p}$	$\text{Var}[X] = \frac{1-p}{p^2}$	$M_X(s) = \frac{pe^s}{1-(1-p)e^s}$
Poisson	$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$	$\mathbb{E}[X] = \lambda$	$\text{Var}[X] = \lambda$	$M_X(s) = e^{\lambda(e^s-1)}$
Gaussian	$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mathbb{E}[X] = \mu$	$\text{Var}[X] = \sigma^2$	$M_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$
Exponential	$f_X(x) = \lambda \exp\{-\lambda x\}$	$\mathbb{E}[X] = \frac{1}{\lambda}$	$\text{Var}[X] = \frac{1}{\lambda^2}$	$M_X(s) = \frac{\lambda}{\lambda - s}$
Uniform	$f_X(x) = \frac{1}{b-a}$	$\mathbb{E}[X] = \frac{a+b}{2}$	$\text{Var}[X] = \frac{(b-a)^2}{12}$	$M_X(s) = \frac{e^{sb}-e^{sa}}{s(b-a)}$

Fourier Transform Table

	$f(t) \longleftrightarrow F(w)$		$f(t) \longleftrightarrow F(w)$
1.	$e^{-at}u(t) \longleftrightarrow \frac{1}{a+jw},\ a>0$	10.	$\text{sinc}^2(\frac{Wt}{2}) \longleftrightarrow \frac{2\pi}{W}\Delta(\frac{w}{2W})$
2.	$e^{at}u(-t) \longleftrightarrow \frac{1}{a-jw},\ a>0$	11.	$e^{-at}\sin(w_0 t)u(t) \longleftrightarrow \frac{w_0}{(a+jw)^2+w_0^2},\ a>0$
3.	$e^{-a\lvert t\rvert} \longleftrightarrow \frac{2a}{a^2+w^2},\ a>0$	12.	$e^{-at}\cos(w_0 t)u(t) \longleftrightarrow \frac{a+jw}{(a+jw)^2+w_0^2},\ a>0$
4.	$\frac{a^2}{a^2+t^2} \longleftrightarrow \pi a e^{-a\lvert w\rvert},\ a>0$	13.	$e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sqrt{2\pi}\sigma e^{-\frac{\sigma^2 w^2}{2}}$
5.	$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+jw)^2},\ a>0$	14.	$\delta(t) \longleftrightarrow 1$
6.	$t^n e^{-at}u(t) \longleftrightarrow \frac{n!}{(a+jw)^{n+1}},\ a>0$	15.	$1 \longleftrightarrow 2\pi\delta(w)$
7.	$\text{rect}(\frac{t}{\tau}) \longleftrightarrow \tau\text{sinc}(\frac{w\tau}{2})$	16.	$\delta(t-t_0) \longleftrightarrow e^{-jwt_0}$
8.	$\text{sinc}(Wt) \longleftrightarrow \frac{\pi}{W}\text{rect}(\frac{w}{2W})$	17.	$e^{jw_0 t} \longleftrightarrow 2\pi\delta(w-w_0)$
9.	$\Delta(\frac{t}{\tau}) \longleftrightarrow \frac{\tau}{2}\text{sinc}^2(\frac{w\tau}{4})$		

Some definitions:

$$\text{sinc}(t) = \frac{\sin(t)}{t} \qquad \text{rect}(t) = \begin{cases} 1, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases} \qquad \Delta(t) = \begin{cases} 1-2\lvert t\rvert, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

Basic Trigonometry

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = 2\cos^2\theta - 1.$$

$$\cos A\cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \qquad \sin A\sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$$
$$\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B)) \qquad \cos A\sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$