ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2017
Instructor: Prof. Stanley H. Chan

Mid Term 1

Fall 2017

Name: ______________________________ PUID: ______________

Please copy and write the following statement:

I certify that I have neither given nor received unauthorized aid on this exam.

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(Please copy and write the above statement.)

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(Signature)

Problem 1. (25 POINTS)
Determine whether the following statements are TRUE or FALSE. (A statement is true if it is always true. Otherwise we will say that the statement is false.) Circle your answer. No partial credit will be given.

1. Let A and B be two events. Then, $\mathbb{P}[A \cup B^c] \leq 1 + \mathbb{P}[A] - \mathbb{P}[B]$.

   TRUE or FALSE.

2. For any discrete random variable X, the expectation $\mathbb{E}[\frac{1}{X}]$ is $\frac{1}{\mathbb{E}[X]}$.

   TRUE or FALSE.

3. Let A and B be two disjoint events. Assume $\mathbb{P}[B] > 0$. Then, $\mathbb{P}[A \mid B] = 0$.

   TRUE or FALSE.

4. The expectation of a random variable cannot be negative.

TRUE or FALSE.

5. Let $0 < p < 1$, then

$$\sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = np.$$

TRUE or FALSE.

Problem 2. (25 POINTS)
Multiple Choice. Please **circle** your answer.

1. Throw a die twice. Let X be the first number and Y be the second number. Find $\mathbb{P}[\min(X, Y) = 3]$.
   (a) 1/6
   (b) 1/9
   (c) 7/36
   (d) 5/ 36
   (e) 1/18
   (f) 1/12
   (g) 1/2
   (h) 1/24
   (i) Problem undefined
   (j) None of the above

2. Let $p_X(k) = c/2^k$, where $k = 2, 3, 4, \ldots$. Find c. (Caution: Note the starting index of k.)
   (a) $\frac{1}{\sqrt{2}}$
   (b) $\sqrt{2}$
   (c) 1
   (d) 1/2
   (e) 2
   (f) 4
   (g) 1/4
   (h) 3/2
   (i) Problem undefined
   (j) None of the above

3. Consider a resistor of $R = 2$ ohms. Let V be the voltage applied to the resistor. Assume that V is a random variable with PMF $p_V(1) = 1/3$, $p_V(2) = 1/4$, $p_V(3) = 1/4$, $p_V(4) = 1/6$. It is given that the power can be calculated using the formula $P = V^2/R$. Find the expectation of P.

   (a) 75/24
   (b) 75/36
   (c) 75/6
   (d) 75/12
   (e) 75/18
   (f) 1
   (g) 2
   (h) 3
   (i) Problem undefined
   (j) None of the above

4. Consider a optical communication system. The data comes at a rate 10^9 bits per second. The error of having one bit of error is 10^{-9}. Find the probability that there are exactly two bits of error. (Hint: Use Poisson approximation.)

   (a) e
   (b) $2e$
   (c) $3e$
   (d) e^{-1}
   (e) $2e^{-1}$
   (f) $e^{-1}/2$
   (g) $e^{-1}/4$
   (h) e^2
   (i) Problem undefined
   (j) None of the above

5. Let $X \sim \text{Poisson}(\lambda)$. Find $\mathbb{E}[(2X+1)^2]$

   (a) $1 + 8\lambda$
   (b) $1 + 4/\lambda + 8/\lambda^2$
   (c) $1 + 4/\lambda + 4/\lambda^2$
   (d) $1 + 4\lambda + 4\lambda^2$
   (e) $1 + 6\lambda + 4\lambda^2$
   (f) $1 + 8\lambda + 4\lambda^2$
   (g) λ
   (h) $\lambda^2 + \lambda$
   (i) Problem undefined
   (j) None of the above

Problem 3. (10 POINTS)
Let A and B be two events such that $\mathbb{P}[A] = \frac{1}{2}$, $\mathbb{P}[B] = \frac{1}{3}$ and $\mathbb{P}[A \cap B] = \frac{1}{5}$.

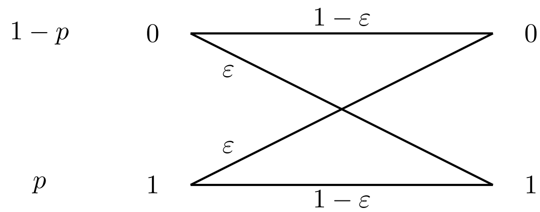
(a) (5 points) Find $\mathbb{P}[A \cup B]$.

$\mathbb{P}[A \cup B] =$

(b) (5 points) Find $\mathbb{P}[B \mid A^c]$.

$\mathbb{P}[B \mid A^c] =$

Problem 4. (20 POINTS)
Consider the following communication channel. A source transmits a string of binary symbols through a noisy communication channel. Each symbol is 0 or 1 with probability $1-p$ and p, respectively, and is received incorrectly with probability ε, respectively. Errors in different symbols transmissions are independent.



Denote S as the source and R as the receiver.

(a) (6 points) What is probability that a 1 is received? (Hint: Find $\mathbb{P}[R=1]$.)

$\mathbb{P}[R=1] =$

(b) (6 points) Condition on the fact that a 1 is received, what is probability that a 1 was sent? (Hint: Find $\mathbb{P}[S=1 \mid R=1]$.)

$\mathbb{P}[S=1 \mid R=1] =$

(c) (8 points) Conditioned on the fact that two 1's are sent, what is probability that a 1 and a 0 are received? That is, find $\mathbb{P}[R = 10 \text{ or } R = 01 \mid S = 11]$.

$\mathbb{P}[R = 10 \text{ or } R = 01 \mid S = 11] =$

Problem 5. (20 POINTS)
Consider two independent random variables X and Y:

$$X = \begin{cases} +1, & \text{with probability } p \\ -1, & \text{with probability } 1-p, \end{cases}$$

and $Y \sim \text{Poisson}(\lambda)$ is a Poisson random variable with parameter λ. Define

$$Z = XY.$$

(a) (5 points) Find $\mathbb{P}[Z = k \mid X = -1]$ and $\mathbb{P}[Z = k \mid X = +1]$, where k is an integer. (Hint: When $X = +1$, $Z = Y$, and when $X = -1$, $Z = -Y$.)

$$\mathbb{P}[Z = k \mid X = -1] = \begin{cases} \qquad\qquad , & k \leq 0 \\ \qquad\qquad , & k > 0 \end{cases}$$

$$\mathbb{P}[Z = k \mid X = +1] = \begin{cases} \qquad\qquad , & k \geq 0 \\ \qquad\qquad , & k < 0 \end{cases}$$

(b) (7 points) Find the PMF of Z. Show your steps using the results in (a). No point if you only write down the answer.

(continue ...)

$$p_Z(k) \overset{\text{def}}{=} \mathbb{P}[Z = k] = \begin{cases} \qquad\qquad\qquad , & k < 0 \\ \qquad\qquad\qquad , & k = 0 \\ \qquad\qquad\qquad , & k > 0 \end{cases}$$

(c) (8 points) Find $\mathbb{E}[Z]$, the expectation of Z. Your steps should follow from the PMF shown in part (b). No point if you only write down the answer.

$\mathbb{E}[Z] =$

Useful Identities

1. $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \ldots = \frac{1}{1-r}$
2. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
3. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$
4. $\sum_{k=1}^{\infty} kr^{k-1} = 1 + 2r + 3r^2 + \ldots = \frac{1}{(1-r)^2}$
5. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^3 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$
6. $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Common Distributions

Bernoulli	$\mathbb{P}[X=1] = p$	$\mathbb{E}[X] = p$	$\text{Var}[X] = p(1-p)$	$M_X(s) = 1 - p + pe^s$
Binomial	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$	$\mathbb{E}[X] = np$	$\text{Var}[X] = np(1-p)$	$M_X(s) = (1 - p + pe^s)^n$
Geometric	$p_X(k) = p(1-p)^{k-1}$	$\mathbb{E}[X] = \frac{1}{p}$	$\text{Var}[X] = \frac{1-p}{p^2}$	$M_X(s) = \frac{pe^s}{1-(1-p)e^s}$
Poisson	$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$	$\mathbb{E}[X] = \lambda$	$\text{Var}[X] = \lambda$	$M_X(s) = e^{\lambda(e^s - 1)}$
Gaussian	$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mathbb{E}[X] = \mu$	$\text{Var}[X] = \sigma^2$	$M_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$
Exponential	$f_X(x) = \lambda \exp\{-\lambda x\}$	$\mathbb{E}[X] = \frac{1}{\lambda}$	$\text{Var}[X] = \frac{1}{\lambda^2}$	$M_X(s) = \frac{\lambda}{\lambda - s}$
Uniform	$f_X(x) = \frac{1}{b-a}$	$\mathbb{E}[X] = \frac{a+b}{2}$	$\text{Var}[X] = \frac{(b-a)^2}{12}$	$M_X(s) = \frac{e^{sb} - e^{sa}}{s(b-a)}$

Fourier Transform Table

	$f(t) \longleftrightarrow F(w)$		$f(t) \longleftrightarrow F(w)$
1.	$e^{-at}u(t) \longleftrightarrow \frac{1}{a+jw},\ a > 0$	10.	$\text{sinc}^2(\frac{Wt}{2}) \longleftrightarrow \frac{2\pi}{W}\Delta(\frac{w}{2W})$
2.	$e^{at}u(-t) \longleftrightarrow \frac{1}{a-jw},\ a > 0$	11.	$e^{-at}\sin(w_0 t)u(t) \longleftrightarrow \frac{w_0}{(a+jw)^2 + w_0^2},\ a > 0$
3.	$e^{-a\|t\|} \longleftrightarrow \frac{2a}{a^2+w^2},\ a > 0$	12.	$e^{-at}\cos(w_0 t)u(t) \longleftrightarrow \frac{a+jw}{(a+jw)^2 + w_0^2},\ a > 0$
4.	$\frac{a^2}{a^2+t^2} \longleftrightarrow \pi a e^{-a\|w\|},\ a > 0$	13.	$e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sqrt{2\pi}\sigma e^{-\frac{\sigma^2 w^2}{2}}$
5.	$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+jw)^2},\ a > 0$	14.	$\delta(t) \longleftrightarrow 1$
6.	$t^n e^{-at}u(t) \longleftrightarrow \frac{n!}{(a+jw)^{n+1}},\ a > 0$	15.	$1 \longleftrightarrow 2\pi\delta(w)$
7.	$\text{rect}(\frac{t}{\tau}) \longleftrightarrow \tau\text{sinc}(\frac{w\tau}{2})$	16.	$\delta(t - t_0) \longleftrightarrow e^{-jwt_0}$
8.	$\text{sinc}(Wt) \longleftrightarrow \frac{\pi}{W}\text{rect}(\frac{w}{2W})$	17.	$e^{jw_0 t} \longleftrightarrow 2\pi\delta(w - w_0)$
9.	$\Delta(\frac{t}{\tau}) \longleftrightarrow \frac{\tau}{2}\text{sinc}^2(\frac{w\tau}{4})$		

Some definitions:

$$\text{sinc}(t) = \frac{\sin(t)}{t} \qquad \text{rect}(t) = \begin{cases} 1, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases} \qquad \Delta(t) = \begin{cases} 1 - 2|t|, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

Basic Trigonometry

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = 2\cos^2\theta - 1.$$

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \quad \sin A \sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$$

$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B)) \quad \cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$

