ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2016
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PURDUE UNIVERSITY

Mid Term 1

Fall 2016

Name: ______________________________ PUID: ______________

Please copy and write the following statement:

I certify that I have neither given nor received unauthorized aid on this exam.

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(Signature)

Problem 1. (25 POINTS)
Determine whether the following statements are TRUE or FALSE. (A statement is true if it is always true. Otherwise we will say that the statement is false.) Circle your answer. No partial credit will be given.

1. For any sets A and B, it holds that $\mathbb{P}[A \cup B] \leq \mathbb{P}[A] + \mathbb{P}[B]$.

   TRUE or FALSE.

2. For any $0 < p < 1$, $\sum_{k=1}^{\infty} p(1-p)^{k-1} = p$.

   TRUE or FALSE.

3. Let A and B be two events. If A and B are independent, then A and B are disjoint. However, if A and B are disjoint, then A and B may not be independent.

   TRUE or FALSE.

4. Let $X \sim \text{Binomial}(n, p)$. Then $\mathbb{E}[(2X-1)^2] = 4n(n-1)p^2$.

TRUE or FALSE.

5. Assume that $\mathbb{P}[A] = 0.3$, $\mathbb{P}[B] = 0.2$ and $\mathbb{P}[C] = 0.1$. It is known that A, B and C are independent, and $A \cap B = \emptyset$. Then the conditional probability $\mathbb{P}[A \cup B \mid C] = 0.5$.

TRUE or FALSE.

Problem 2. (25 POINTS)
Multiple Choice. Please **circle** your answer.

1. Assume that $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$ and $\mathbb{P}[C] > 0$. Then, $\mathbb{P}[A \cap B \cap C] =$

   (a) $\mathbb{P}[A|B]\,\mathbb{P}[B|C]\,\mathbb{P}[C|A]$
   (b) $\mathbb{P}[B]\,\mathbb{P}[A|B]\,\mathbb{P}[C|A \cap B]$
   (c) $\mathbb{P}[A|B]\,\mathbb{P}[B]\,\mathbb{P}[A]\,\mathbb{P}[C|A \cap B]$
   (d) $\mathbb{P}[C|B \cap A]\,\mathbb{P}[B|A]\,\mathbb{P}[A]$
   (e) (a) and (b)
   (f) (a) and (b)
   (g) (b) and (c)
   (h) (b) and (d)
   (i) (a) and (b) and (d)
   (j) All of the above
   (k) None of the above

2. Let $p_X(k) = c/2^k$ for $k = 2, 3, \ldots$. Then, $c =$

   (a) 1
   (b) 2
   (c) 4
   (d) $\frac{1}{2}$
   (e) $\frac{1}{4}$
   (f) $\frac{1}{8}$
   (g) $\frac{2}{3}$
   (h) $\frac{3}{2}$
   (i) $\frac{3}{4}$
   (j) $\frac{4}{3}$

3. A company makes three types of pens: Type A, Type B and Type C. Among all the pens, 40% are Type A, 30% are Type B, and 30% are Type C. The probability that Type A has defect is 0.2, that Type B has defect is 0.1, and that Type C has defect is 0.1. Suppose that you get a pen and see that it is defective. Find the probability that the pen was Type A.

   (a) 0.4
   (b) 0.2
   (c) 0.1
   (d) 0.16
   (e) 0.14
   (f) 0.12
   (g) 0.08
   (h) 0.04
   (i) 0.02
   (j) None of the above

4. Throw a dice twice. Let X be the absolute difference in the number of dots facing up. Find $\mathbb{P}[X \leq 2]$.

   (a) 1/8
   (b) 1/6
   (c) 1/4
   (d) 1/3
   (e) 1/2
   (f) 3/4
   (g) 2/3
   (h) 5/8
   (i) 7/12
   (j) None of the above

5. Let $p_X(k) = \sin\left(\frac{\pi}{2}k\right)$ for $k = 0, 1, 2, 3$. The expected value $\mathbb{E}[X]$ is

   (a) 0
   (b) 1
   (c) 2
   (d) $\frac{1}{4}$
   (e) $\frac{1}{2}$
   (f) -1
   (g) -2
   (h) $-\frac{1}{4}$
   (i) $-\frac{1}{2}$
   (j) None of the above

Problem 3. (30 POINTS)
Let A and B be two events such that $\mathbb{P}[A \mid B] = 0.4$, $\mathbb{P}[B \mid A] = 0.3$ and $\mathbb{P}[A] = 0.5$.

(a) (5 points) State clearly the definition of two events A and B being independent.

(b) (10 points) Find $\mathbb{P}[A \cap B]$ and $\mathbb{P}[A \cup B]$.

(c) (7 points) Find $\mathbb{P}[A \cup B^c]$ and $\mathbb{P}[A^c \cup B]$.

(d) (8 points) Find $\mathbb{P}[A \cup B^c \mid A^c \cup B]$, i.e., the conditional probability of $A \cup B^c$ given $A^c \cup B$.

Problem 4. (20 POINTS)
Consider three random variables $X \sim \text{Bernoulli}(p)$, $Z_1 \sim \text{Poisson}(\lambda_1)$, and $Z_2 \sim \text{Poisson}(\lambda_2)$. Let Y be a random variable such that

$$Y = \begin{cases} Z_1, & \text{if} \quad X = 1, \\ Z_2, & \text{if} \quad X = 0. \end{cases}$$

(a) (10 points) Find the PMF of Y.

(b) (10 points) Find the expectation of Y.

Useful Identities

1. $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \ldots = \frac{1}{1-r}$
2. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
3. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$
4. $\sum_{k=1}^{\infty} kr^{k-1} = 1 + 2r + 3r^2 + \ldots = \frac{1}{(1-r)^2}$
5. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^3 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$
6. $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Common Distributions

Bernoulli	$\mathbb{P}[X=1] = p$	$\mathbb{E}[X] = p$	$\text{Var}[X] = p(1-p)$	$M_X(s) = 1 - p + pe^s$
Binomial	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$	$\mathbb{E}[X] = np$	$\text{Var}[X] = np(1-p)$	$M_X(s) = (1 - p + pe^s)^n$
Geometric	$p_X(k) = p(1-p)^{k-1}$	$\mathbb{E}[X] = \frac{1}{p}$	$\text{Var}[X] = \frac{1-p}{p^2}$	$M_X(s) = \frac{pe^s}{1-(1-p)e^s}$
Poisson	$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$	$\mathbb{E}[X] = \lambda$	$\text{Var}[X] = \lambda$	$M_X(s) = e^{\lambda(e^s-1)}$
Gaussian	$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mathbb{E}[X] = \mu$	$\text{Var}[X] = \sigma^2$	$M_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$
Exponential	$f_X(x) = \lambda \exp\{-\lambda x\}$	$\mathbb{E}[X] = \frac{1}{\lambda}$	$\text{Var}[X] = \frac{1}{\lambda^2}$	$M_X(s) = \frac{\lambda}{\lambda - s}$
Uniform	$f_X(x) = \frac{1}{b-a}$	$\mathbb{E}[X] = \frac{a+b}{2}$	$\text{Var}[X] = \frac{(b-a)^2}{12}$	$M_X(s) = \frac{e^{sb} - e^{sa}}{s(b-a)}$

Fourier Transform Table

	$f(t) \longleftrightarrow F(w)$		$f(t) \longleftrightarrow F(w)$
1.	$e^{-at}u(t) \longleftrightarrow \frac{1}{a+jw},\ a>0$	10.	$\text{sinc}^2(\frac{Wt}{2}) \longleftrightarrow \frac{2\pi}{W}\Delta(\frac{w}{2W})$
2.	$e^{at}u(-t) \longleftrightarrow \frac{1}{a-jw},\ a>0$	11.	$e^{-at}\sin(w_0 t)u(t) \longleftrightarrow \frac{w_0}{(a+jw)^2+w_0^2},\ a>0$
3.	$e^{-a\|t\|} \longleftrightarrow \frac{2a}{a^2+w^2},\ a>0$	12.	$e^{-at}\cos(w_0 t)u(t) \longleftrightarrow \frac{a+jw}{(a+jw)^2+w_0^2},\ a>0$
4.	$\frac{a^2}{a^2+t^2} \longleftrightarrow \pi a e^{-a\|w\|},\ a>0$	13.	$e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sqrt{2\pi}\sigma e^{-\frac{\sigma^2 w^2}{2}}$
5.	$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+jw)^2},\ a>0$	14.	$\delta(t) \longleftrightarrow 1$
6.	$t^n e^{-at}u(t) \longleftrightarrow \frac{n!}{(a+jw)^{n+1}},\ a>0$	15.	$1 \longleftrightarrow 2\pi\delta(w)$
7.	$\text{rect}(\frac{t}{\tau}) \longleftrightarrow \tau\text{sinc}(\frac{w\tau}{2})$	16.	$\delta(t-t_0) \longleftrightarrow e^{-jwt_0}$
8.	$\text{sinc}(Wt) \longleftrightarrow \frac{\pi}{W}\text{rect}(\frac{w}{2W})$	17.	$e^{jw_0 t} \longleftrightarrow 2\pi\delta(w-w_0)$
9.	$\Delta(\frac{t}{\tau}) \longleftrightarrow \frac{\tau}{2}\text{sinc}^2(\frac{w\tau}{4})$		

Some definitions:

$$\text{sinc}(t) = \frac{\sin(t)}{t} \qquad \text{rect}(t) = \begin{cases} 1, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases} \qquad \Delta(t) = \begin{cases} 1 - 2|t|, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

Basic Trigonometry

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = 2\cos^2\theta - 1.$$

$$\cos A\cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \qquad \sin A\sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$$
$$\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B)) \qquad \cos A\sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$