Final
Fall 2017

Name: _______________________________ PUID: ________________

Please copy and write the following statement:

*I certify that I have neither given nor received unauthorized aid on this exam.*

(Please copy and write the above statement.)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Score</th>
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<tbody>
<tr>
<td>Q1</td>
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<td>Q2</td>
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Problem 1. (10 points)
Determine whether the following statements are TRUE or FALSE. (A statement is true if it is always true. Otherwise it is false.) Circle your answer. No partial credit will be given.

(a) If \( P(A \cap B) = P(A)P(B) \), then \( A \) and \( B \) are disjoint.

\[ \text{TRUE or FALSE.} \]

(b) Let \( X \) be a random variable with PMF

\[ p_X(k) = P(X = k) = \left( \frac{1}{2} \right)^k, \quad k = 0, 1, 2, \ldots. \]

Then \( p_X(k) \) is a valid PMF.

\[ \text{TRUE or FALSE.} \]

(c) Let \( X \sim \text{Binomial}(n, p) \), \( n = 1000 \), \( p = \frac{1}{1000} \). Then

\[ P(X = 2) \approx e^{-1}. \]

\[ \text{TRUE or FALSE.} \]

(d) If \( X \) and \( Y \) are independent Gaussian random variables, then

\[ Z = aX + bY \]

is also a Gaussian random variable, where \( a \) and \( b \) are constants.

\[ \text{TRUE or FALSE.} \]

(e) Let \( X \) be a random variable with moment generating function \( M_X(s) \). Then,

\[ \mathbb{E}[X^2] = \frac{d^2}{ds^2} M_X(s) \bigg|_{s=0}. \]

\[ \text{TRUE or FALSE.} \]
(f) Let $X$ be a random variable, and let $y$ and $z$ be two constants. If $y \geq z$, then,

$$P[X \leq y] \leq P[X \leq z].$$

TRUE or FALSE.

(g) Let $X \sim \mathcal{N}(0, 1)$, and let $Y \mid X \sim \mathcal{N}(X, 1)$. Then,

$$E[Y] = 0.$$

TRUE or FALSE.

(h) Let $X_1, \ldots, X_n$ be iid Bernoulli with $P[X_i = 1] = p$ for all $i$. Let $Y = \sum_{i=1}^{n} X_i$. Then,

$$Y \xrightarrow{d} \mathcal{N}(np, np(1-p)).$$

TRUE or FALSE.

(i) If $\text{Cov}(X, Y) = 0$, then

$$E[(X + Y)^2] = E[X^2] + E[Y^2]$$

TRUE or FALSE.

(j) If $X(t)$ is wide sense stationary, then

$$E[X(t)] = 0, \quad \text{for all } t.$$ 

TRUE or FALSE.
Problem 2. (15 POINTS)
Circle one and only one answer. If I cannot tell which answer you are circling, I will give you a zero. There is no partial credit for this problem.

1. Consider a pair of random variables $X$ and $Y$ with joint PMF shown below.

\[
\begin{array}{c|cccc}
X = 1 & 1 & 2 & 3 & 4 \\
1 & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} \\
2 & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} \\
3 & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} \\
4 & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20}
\end{array}
\]

Find $P[X = 2 \mid Y = 3]$

(a) $\frac{1}{2}$
(b) $\frac{1}{4}$
(c) $\frac{1}{6}$
(d) $\frac{1}{10}$
(e) $\frac{1}{12}$
(f) $\frac{1}{20}$
(g) $\frac{3}{20}$
(h) $\frac{7}{20}$
(i) None of the above
(j) Problem undefined

2. Let $X \sim \mathcal{N}(0, 1)$. Let

\[Y = \begin{cases} 
X, & X \geq 0, \\
0, & X < 0.
\end{cases}\]

Find the CDF of $Y$.

Remark: There is no option (i) and (j) for this problem.
3. Let $X \sim N(\mu, \sigma^2)$. Let $Z = \sqrt{|X|}$. Find $F_Z(z)$.
   
   (a) $\Phi \left( \frac{z-\mu}{\sigma} \right) - \Phi \left( -\frac{z+\mu}{\sigma} \right)$
   
   (b) $\Phi \left( -\frac{z+\mu}{\sigma} \right) - \Phi \left( \frac{z-\mu}{\sigma} \right)$
   
   (c) $\Phi \left( \frac{z^2-\mu^2}{\sigma} \right) - \Phi \left( -\frac{z^2+\mu^2}{\sigma} \right)$
   
   (d) $\Phi \left( -\frac{z^2+\mu^2}{\sigma} \right) - \Phi \left( \frac{z^2-\mu^2}{\sigma} \right)$
   
   (e) $\Phi \left( \frac{\sqrt{z^2-\mu^2}}{\sigma} \right) - \Phi \left( -\frac{\sqrt{z^2+\mu^2}}{\sigma} \right)$
   
   (f) $\Phi \left( -\frac{\sqrt{z^2+\mu^2}}{\sigma} \right) - \Phi \left( \frac{\sqrt{z^2-\mu^2}}{\sigma} \right)$
   
   (i) Problem undefined.
   
   (j) None of the above.

4. Let $X_1, \ldots, X_N$ be a sequence of iid random variables with $\mathbb{E}[X_i] = \mu$ for all $i$. The number of random variables $N$, however, is also random. Its PMF is
   
   $\mathbb{P}[N = n] = \left( \frac{1}{2} \right)^n$, \quad n = 1, 2, \ldots.
   
   Let $Z_N = X_1 + \ldots + X_N$. Find $\mathbb{E}[Z_N]$. (Hint: First find the conditional expectation $\mathbb{E}[Z_N \mid N = n]$.)
   
   (a) $\mu$
   
   (b) $2\mu$
   
   (c) $4\mu$
   
   (d) $8\mu$
   
   (e) $N\mu$
   
   (f) $\mu/N$
   
   (g) $\mu/2^N$
   
   (h) $2^N \mu$
   
   (i) Problem undefined.
   
   (j) None of the above.
5. Let $X_1, \ldots, X_N$ be a sequence of iid Poisson random variables with $X_i \sim \text{Poisson}(\lambda)$. Assume $\lambda = 1$ and $N = 100$. Let

$$Z = \frac{1}{N} \sum_{i=1}^{N} X_i.$$ 

Use Central Limit Theorem to approximate the probability

$$P \left[ \frac{\lambda}{2} \leq Z \leq \frac{3\lambda}{2} \right].$$

(a) $\Phi(1) - \Phi(-1)$
(b) $\Phi(2) - \Phi(-2)$
(c) $\Phi(3) - \Phi(-3)$
(d) $\Phi(5) - \Phi(-5)$
(e) $\Phi(10) - \Phi(-10)$
(f) $\Phi(100) - \Phi(-100)$
(g) $\Phi(\sqrt{2}) - \Phi(-\sqrt{2})$
(h) $\Phi(\sqrt{5}) - \Phi(-\sqrt{5})$
(i) Problem undefined
(j) None of the above.
Problem 3. (15 points)
Consider two random variables $X$ and $Y$ with PDFs

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}, \quad f_Y(y) = \begin{cases} 1, & -\frac{1}{2} \leq y \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Let $W = X + Y$ and $Z = X - Y$. Find $\text{Cov}(W, Z)$.

$$\text{Cov}(W, Z) =$$
Problem 4. (15 points)
Let $X \sim \text{Uniform}[1, 2]$, and $Y \mid X \sim \mathcal{N}(0, X^2)$. That is, the conditional distribution of $Y$ given $X$ is a Gaussian with mean 0 and variance $X^2$. Find $\mathbb{E}[Xe^Y]$. 

\[ \mathbb{E}[Xe^Y] = \]
Problem 5. (20 points)
Let $U(t)$ be a zero mean random process with $R_U(t, s) = \min(t, s)$. Define

$$X(t) = e^{tU(e^{-2t})}.$$ 

(a) (10 points) Find $R_X(\tau)$. Show your steps clearly or otherwise no point.

$$R_X(\tau) =$$
(b) (10 points) Suppose that $X(t)$ passes through an LTI system below. Find $R_Y(\tau)$.

\[ R_Y(\tau) = \]

\[ \frac{d}{dt} \]

\[ + \]

\[ Y(t) \]
(c) (5 points) Find $R_{Y,X}(\tau)$. 

$$R_{Y,X}(\tau) =$$
Problem 6. (20 points)
Let $X_1, \ldots, X_N$ be a sequence of iid random variables with $X_i \sim \mathcal{N}(\theta, \sigma^2)$. Assume that $\sigma$ is known and fixed, and $\theta$ is unknown.

(a) (5 points) Suppose that we observed $N$ realizations $x_1, \ldots, x_N$, find the likelihood function.

\[
f_{X_1, \ldots, X_N}(x_1, \ldots, x_N \mid \theta) =
\]

(b) (7 points) Find the maximum likelihood estimate of $\theta$. Show your steps clearly or otherwise no point.

\[
\hat{\theta}_{\text{ML}} =
\]
(b) (8 points) Suppose that we observed $N$ realizations $x_1, \ldots, x_N$. Let us also assume that $\theta$ has a prior distribution with PDF $f_\Theta(\theta) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{(\theta - \mu)^2}{2} \right\}$. Find the maximum-a-posteriori estimate of $\theta$.

\[ \hat{\theta}_\text{MAP} = \]
## Useful Identities

1. \[ \sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \ldots = \frac{1}{1-r} \quad r < 1 \]
2. \[ \sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \]
3. \[ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots \]
4. \[ \sum_{k=1}^{\infty} k^{-1} = 1 + 2 + 3 + \ldots = \frac{1}{(1-r)^2} \]
5. \[ \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \]
6. \[ (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \]

## Common Distributions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>PDF ( f_X(x) )</th>
<th>CDF ( F_X(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bernoulli</td>
<td>( f_X(x) = \binom{n}{k} p^k (1-p)^{n-k} )</td>
<td>( F_X(x) = \binom{x}{k} p^k (1-p)^{n-k} )</td>
</tr>
<tr>
<td>Binomial</td>
<td>( f_X(k) = \binom{n}{k} \frac{1}{2^n} )</td>
<td>( F_X(x) = \frac{x}{n+1} )</td>
</tr>
<tr>
<td>Geometric</td>
<td>( f_X(x) = \theta e^{-\theta x} )</td>
<td>( F_X(x) = 1 - e^{-\theta x} )</td>
</tr>
<tr>
<td>Poisson</td>
<td>( f_X(x) = \frac{\lambda^x e^{-\lambda}}{x!} )</td>
<td>( F_X(x) = e^{-\lambda} )</td>
</tr>
<tr>
<td>Gaussian</td>
<td>( f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} )</td>
<td>( F_X(x) = \frac{1}{2} )</td>
</tr>
<tr>
<td>Exponential</td>
<td>( f_X(x) = \lambda e^{-\lambda x} )</td>
<td>( F_X(x) = \frac{x}{\lambda} )</td>
</tr>
<tr>
<td>Uniform</td>
<td>( f_X(x) = \frac{1}{b-a} )</td>
<td>( F_X(x) = \frac{x-a}{b-a} )</td>
</tr>
</tbody>
</table>

## Fourier Transform Table

<table>
<thead>
<tr>
<th>Time Domain</th>
<th>Frequency Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>( F(w) )</td>
</tr>
<tr>
<td>( e^{-at} u(t) )</td>
<td>( \frac{w}{a+jw} )</td>
</tr>
<tr>
<td>( e^{at} u(-t) )</td>
<td>( \frac{w}{a-jw} )</td>
</tr>
<tr>
<td>( e^{-at}</td>
<td>t</td>
</tr>
<tr>
<td>( e^{-\frac{t^2}{2\sigma^2}} )</td>
<td>( \sqrt{2\pi\sigma} e^{-\frac{w^2}{2\sigma^2}} )</td>
</tr>
<tr>
<td>( t^n e^{-at} u(t) )</td>
<td>( \frac{n!}{(a+jw)^n} )</td>
</tr>
<tr>
<td>( \text{rect}(\frac{t}{\lambda}) )</td>
<td>( \frac{\pi}{\lambda} \text{rect}(\frac{w}{2\pi\lambda}) )</td>
</tr>
<tr>
<td>( \Delta(\frac{t}{\lambda}) )</td>
<td>( \lambda \text{rect}(\frac{w}{2\pi\lambda}) )</td>
</tr>
</tbody>
</table>

### Some definitions:

\[
\text{sinc}(t) = \frac{\sin(t)}{t}, \quad \text{rect}(t) = \begin{cases} 1, & -0.5 \leq t \leq 0.5, \\ 0, & \text{otherwise} \end{cases}, \quad \Delta(t) = \begin{cases} 1 - 2|t|, & -0.5 \leq t \leq 0.5, \\ 0, & \text{otherwise} \end{cases}
\]

## Basic Trigonometry

\[
e^{j\theta} = \cos \theta + j \sin \theta, \quad \sin 2\theta = 2 \sin \theta \cos \theta, \quad \cos 2\theta = 2 \cos^2 \theta - 1.
\]

\[
\begin{align*}
\cos A \cos B &= \frac{1}{2}(\cos(A + B) + \cos(A - B)) \\
\sin A \sin B &= \frac{1}{2}(\cos(A + B) - \cos(A - B)) \\
\sin A \cos B &= \frac{1}{2}(\sin(A + B) + \sin(A - B)) \\
\cos A \sin B &= \frac{1}{2}(\sin(A + B) - \sin(A - B))
\end{align*}
\]