Final

Fall 2017

Name: ______________________________ PUID: ______________

Please copy and write the following statement:

I certify that I have neither given nor received unauthorized aid on this exam.

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(Signature)

Problem	Score
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Total	

Problem 1. (10 POINTS)
Determine whether the following statements are TRUE or FALSE. (A statement is true if it is *always* true. Otherwise it is false.) Circle your answer. No partial credit will be given.

(a) If $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$, then A and B are disjoint.

TRUE or FALSE.

(b) Let X be a random variable with PMF

$$p_X(k) = \mathbb{P}[X = k] = \left(\frac{1}{2}\right)^k, \qquad k = 0, 1, 2, \ldots.$$

Then $p_X(k)$ is a valid PMF.

TRUE or FALSE.

(c) Let $X \sim \text{Binomial}(n, p)$, $n = 1000$, $p = \frac{1}{1000}$. Then

$$\mathbb{P}[X = 2] \approx \frac{e^{-1}}{2}.$$

TRUE or FALSE.

(d) If X and Y are independent Gaussian random variables, then

$$Z = aX + bY$$

is also a Gaussian random variable, where a and b are constants.

TRUE or FALSE.

(e) Let X be a random variable with moment generating function $M_X(s)$. Then,

$$\mathbb{E}[X^2] = \frac{d^2}{ds^2} M_X(s)\Big|_{s=0}.$$

TRUE or FALSE.

(f) Let X be a random variable, and let y and z be two constants. If $y \geq z$, then,

$$\mathbb{P}[X \leq y] \leq \mathbb{P}[X \leq z].$$

TRUE or FALSE.

(g) Let $X \sim \mathcal{N}(0,1)$, and let $Y \,|\, X \sim \mathcal{N}(X,1)$. Then,

$$\mathbb{E}[Y] = 0.$$

TRUE or FALSE.

Not Required (h) Let $X_1, \ldots, X_n$ be iid Bernoulli with $\mathbb{P}[X_i = 1] = p$ for all i. Let $Y = \sum_{i=1}^n X_i$. Then,

$$Y \xrightarrow{d} \mathcal{N}(np, np(1-p)).$$

TRUE or FALSE.

(i) If $\text{Cov}(X,Y) = 0$, then

$$\mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2] + \mathbb{E}[Y^2]$$

TRUE or FALSE.

(j) If $X(t)$ is wide sense stationary, then

$$\mathbb{E}[X(t)] = 0, \qquad \text{for all } t.$$

TRUE or FALSE.

Problem 2. (15 POINTS)
Circle one and only one answer. If I cannot tell which answer you are circling, I will give you a zero. There is no partial credit for this problem.

1. Consider a pair of random variables X and Y with joint PMF shown below.

	Y= 1	2	3	4
X = 1	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{0}{20}$
2	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$
3	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{0}{20}$
4	$\frac{0}{20}$	$\frac{3}{20}$	$\frac{1}{20}$	$\frac{2}{20}$

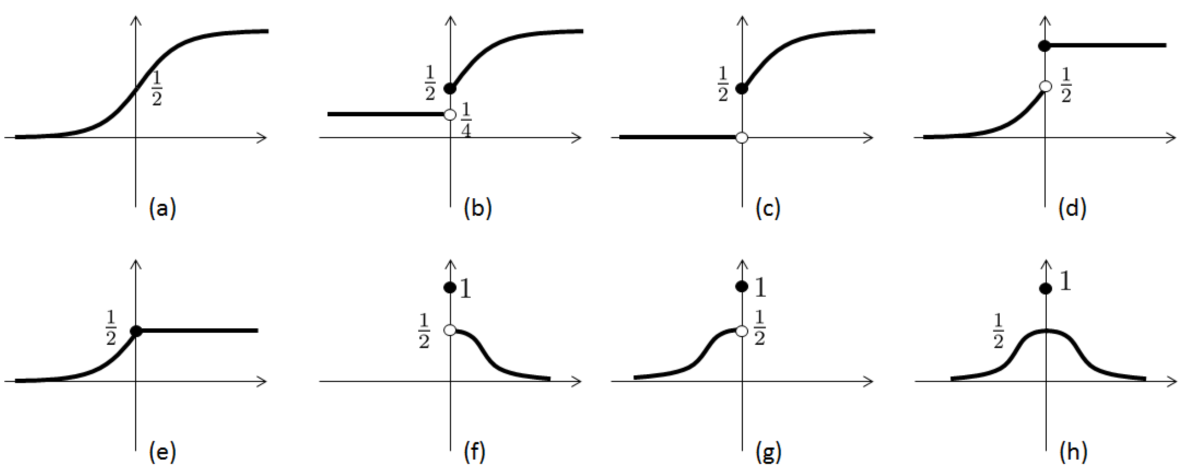
Find $\mathbb{P}[X = 2 \,|\, Y = 3]$

(a) $\frac{1}{2}$
(b) $\frac{1}{4}$
(c) $\frac{1}{6}$
(d) $\frac{1}{10}$
(e) $\frac{1}{12}$
(f) $\frac{1}{20}$
(g) $\frac{3}{20}$
(h) $\frac{7}{20}$
(i) None of the above
(j) Problem undefined

2. Let $X \sim \mathcal{N}(0, 1)$. Let

$$Y = \begin{cases} X, & X \geq 0, \\ 0, & X < 0. \end{cases}$$

Find the CDF of Y.



Remark: There is no option (i) and (j) for this problem.

3. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Let $Z = \sqrt{|X|}$. Find $F_Z(z)$.

   (a) $\Phi\left(\frac{z-\mu}{\sigma}\right) - \Phi\left(-\frac{z+\mu}{\sigma}\right)$
   (b) $\Phi\left(-\frac{z+\mu}{\sigma}\right) - \Phi\left(\frac{z-\mu}{\sigma}\right)$
   (c) $\Phi\left(\frac{z^2-\mu^2}{\sigma}\right) - \Phi\left(-\frac{z^2+\mu^2}{\sigma}\right)$
   (d) $\Phi\left(-\frac{z^2+\mu^2}{\sigma}\right) - \Phi\left(\frac{z^2-\mu^2}{\sigma}\right)$
   (e) $\Phi\left(\frac{z^2-\mu}{\sigma}\right) - \Phi\left(-\frac{z^2+\mu}{\sigma}\right)$
   (f) $\Phi\left(-\frac{z^2+\mu}{\sigma}\right) - \Phi\left(\frac{z^2-\mu}{\sigma}\right)$
   (g) $\Phi\left(\frac{\sqrt{z}-\mu}{\sigma}\right) - \Phi\left(-\frac{\sqrt{z}+\mu}{\sigma}\right)$
   (h) $\Phi\left(-\frac{\sqrt{z}+\mu}{\sigma}\right) - \Phi\left(\frac{\sqrt{z}-\mu}{\sigma}\right)$
   (i) Problem undefined
   (j) None of the above

4. Let $X_1, \ldots, X_N$ be a sequence of iid random variables with $\mathbb{E}[X_i] = \mu$ for all i. The number of random variables N, however, is also random. Its PMF is

$$\mathbb{P}[N = n] = \left(\frac{1}{2}\right)^n, \qquad n = 1, 2, \ldots.$$

   Let $Z_N = X_1 + \ldots + X_N$. Find $\mathbb{E}[Z_N]$. (Hint: First find the conditional expectation $\mathbb{E}[Z_N \mid N = n]$.)

   (a) μ
   (b) 2μ
   (c) 4μ
   (d) 8μ
   (e) $N\mu$
   (f) μ/N
   (g) $\mu/2^N$
   (h) $2^N\mu$
   (i) Problem undefined.
   (j) None of the above.

Not Required 5. Let $X_1, \ldots, X_N$ be a sequence of iid Poisson random variables with $X_i \sim \text{Poisson}(\lambda)$. Assume $\lambda = 1$ and $N = 100$. Let

$$Z = \frac{1}{N} \sum_{i=1}^{N} X_i.$$

Use Central Limit Theorem to approximate the probability

$$\mathbb{P}\left[\frac{\lambda}{2} \leq Z \leq \frac{3\lambda}{2}\right].$$

(a) $\Phi(1) - \Phi(-1)$

(b) $\Phi(2) - \Phi(-2)$

(c) $\Phi(3) - \Phi(-3)$

(d) $\Phi(5) - \Phi(-5)$

(e) $\Phi(10) - \Phi(-10)$

(f) $\Phi(100) - \Phi(-100)$

(g) $\Phi(\sqrt{2}) - \Phi(-\sqrt{2})$

(h) $\Phi(\sqrt{5}) - \Phi(-\sqrt{5})$

(i) Problem undefined

(j) None of the above.

Problem 3. (15 POINTS)
Consider two random variables X and Y with PDFs

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}, \qquad f_Y(y) = \begin{cases} 1, & -\frac{1}{2} \leq y \leq \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Let $W = X + Y$ and $Z = X - Y$. Find $\text{Cov}(W, Z)$.

$\text{Cov}(W, Z) =$

Problem 4. (15 POINTS)
Let $X \sim \text{Uniform}[1,2]$, and $Y \,|\, X \sim \mathcal{N}(0, X^2)$. That is, the conditional distribution of Y given X is a Gaussian with mean 0 and variance X^2. Find $\mathbb{E}[Xe^Y]$.

$\mathbb{E}[Xe^Y] =$

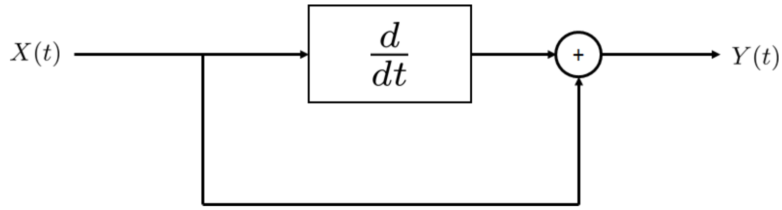
Problem 5. (20 POINTS)
Let $U(t)$ be a zero mean random process with $R_U(t, s) = \min(t, s)$. Define

$$X(t) = e^t U\left(e^{-2t}\right).$$

(a) (10 points) Find $R_X(\tau)$. Show your steps clearly or otherwise no point.

$R_X(\tau) =$

(b) (10 points) Suppose that $X(t)$ passes through an LTI system below. Find $R_Y(\tau)$.



$R_Y(\tau) =$

(c) (5 points) Find $R_{Y,X}(\tau)$.

$R_{Y,X}(\tau) =$

Problem 6. (20 POINTS)
Let $X_1, \ldots, X_N$ be a sequence of iid random variables with $X_i \sim \mathcal{N}(\theta, \sigma^2)$. Assume that σ is known and fixed, and θ is unknown.

Not Required (a) (5 points) Suppose that we observed N realizations $x_1, \ldots, x_N$, find the likelihood function.

$f_{X_1,\ldots,X_N}(x_1, \ldots, x_N \mid \theta) =$

Not Required (b) (7 points) Find the maximum likelihood estimate of θ. Show your steps clearly or otherwise no point.

$\widehat{\theta}_{\text{ML}} =$

Not Required (b) (8 points) Suppose that we observed N realizations $x_1, \ldots, x_N$. Let us also assume that θ has a prior distribution with PDF $f_\Theta(\theta) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(\theta-\mu)^2}{2}\right\}$. Find the maximum-a-posteriori estimate of θ.

$\widehat{\theta}_{\text{MAP}} =$

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Useful Identities

1. $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \ldots = \frac{1}{1-r}$
2. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
3. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$
4. $\sum_{k=1}^{\infty} kr^{k-1} = 1 + 2r + 3r^2 + \ldots = \frac{1}{(1-r)^2}$
5. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^3 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$
6. $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Common Distributions

Bernoulli	$\mathbb{P}[X=1] = p$	$\mathbb{E}[X] = p$	$\text{Var}[X] = p(1-p)$	$M_X(s) = 1 - p + pe^s$
Binomial	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$	$\mathbb{E}[X] = np$	$\text{Var}[X] = np(1-p)$	$M_X(s) = (1-p+pe^s)^n$
Geometric	$p_X(k) = p(1-p)^{k-1}$	$\mathbb{E}[X] = \frac{1}{p}$	$\text{Var}[X] = \frac{1-p}{p^2}$	$M_X(s) = \frac{pe^s}{1-(1-p)e^s}$
Poisson	$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$	$\mathbb{E}[X] = \lambda$	$\text{Var}[X] = \lambda$	$M_X(s) = e^{\lambda(e^s-1)}$
Gaussian	$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mathbb{E}[X] = \mu$	$\text{Var}[X] = \sigma^2$	$M_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$
Exponential	$f_X(x) = \lambda \exp\{-\lambda x\}$	$\mathbb{E}[X] = \frac{1}{\lambda}$	$\text{Var}[X] = \frac{1}{\lambda^2}$	$M_X(s) = \frac{\lambda}{\lambda - s}$
Uniform	$f_X(x) = \frac{1}{b-a}$	$\mathbb{E}[X] = \frac{a+b}{2}$	$\text{Var}[X] = \frac{(b-a)^2}{12}$	$M_X(s) = \frac{e^{sb}-e^{sa}}{s(b-a)}$

Fourier Transform Table

	$f(t) \longleftrightarrow F(w)$		$f(t) \longleftrightarrow F(w)$
1.	$e^{-at}u(t) \longleftrightarrow \frac{1}{a+jw},\ a>0$	10.	$\text{sinc}^2(\frac{Wt}{2}) \longleftrightarrow \frac{2\pi}{W}\Delta(\frac{w}{2W})$
2.	$e^{at}u(-t) \longleftrightarrow \frac{1}{a-jw},\ a>0$	11.	$e^{-at}\sin(w_0 t)u(t) \longleftrightarrow \frac{w_0}{(a+jw)^2+w_0^2},\ a>0$
3.	$e^{-a\|t\|} \longleftrightarrow \frac{2a}{a^2+w^2},\ a>0$	12.	$e^{-at}\cos(w_0 t)u(t) \longleftrightarrow \frac{a+jw}{(a+jw)^2+w_0^2},\ a>0$
4.	$\frac{a^2}{a^2+t^2} \longleftrightarrow \pi a e^{-a\|w\|},\ a>0$	13.	$e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sqrt{2\pi}\sigma e^{-\frac{\sigma^2 w^2}{2}}$
5.	$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+jw)^2},\ a>0$	14.	$\delta(t) \longleftrightarrow 1$
6.	$t^n e^{-at}u(t) \longleftrightarrow \frac{n!}{(a+jw)^{n+1}},\ a>0$	15.	$1 \longleftrightarrow 2\pi\delta(w)$
7.	$\text{rect}(\frac{t}{\tau}) \longleftrightarrow \tau\text{sinc}(\frac{w\tau}{2})$	16.	$\delta(t-t_0) \longleftrightarrow e^{-jwt_0}$
8.	$\text{sinc}(Wt) \longleftrightarrow \frac{\pi}{W}\text{rect}(\frac{w}{2W})$	17.	$e^{jw_0 t} \longleftrightarrow 2\pi\delta(w-w_0)$
9.	$\Delta(\frac{t}{\tau}) \longleftrightarrow \frac{\tau}{2}\text{sinc}^2(\frac{w\tau}{4})$		

Some definitions:

$$\text{sinc}(t) = \frac{\sin(t)}{t} \qquad \text{rect}(t) = \begin{cases} 1, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases} \qquad \Delta(t) = \begin{cases} 1 - 2|t|, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

Basic Trigonometry

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = 2\cos^2\theta - 1.$$

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \qquad \sin A \sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$$

$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B)) \qquad \cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$