ECE 302: Lecture A.9 Cross-correlation through LTI Systems

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The missing parts of our story

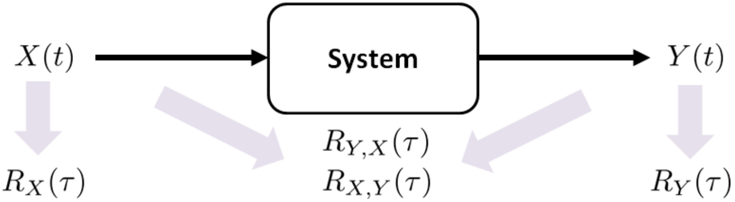

Figure: The source of the signals when defining $R_X(\tau)$, $R_{X,Y}(\tau)$, $R_{Y,X}(\tau)$ and $R_Y(\tau)$.

Jointly WSS

Definition

Two random processes $X(t)$ and $Y(t)$ are **jointly WSS** if

1. $X(t)$ is WSS and $Y(t)$ is WSS
2. $R_{X,Y}(t_1, t_2) = \mathbb{E}[X(t_1)Y(t_2)]$ is a function of $t_1 - t_2$.

If $X(t)$ and $Y(t)$ are jointly WSS, then we write

$$R_{X,Y}(t_1, t_2) = R_{X,Y}(\tau) \stackrel{\text{def}}{=} \mathbb{E}\left[X(t+\tau)Y(\tau)\right].$$

$R_{X,Y}(\tau) = R_{Y,X}(-\tau)$

Lemma

For any random processes $X(t)$ and $Y(t)$, the cross-correlation $R_{X,Y}(\tau)$ is related to $R_{Y,X}(\tau)$ as

$$R_{X,Y}(\tau) = R_{Y,X}(-\tau). \tag{1}$$

Proof:

$$\begin{aligned} R_{Y,X}(-\tau) &= \mathbb{E}\left[Y(t-\tau)X(t)\right] \\ &= \mathbb{E}\left[X(t)Y(t-\tau)\right] \\ &= \mathbb{E}\left[X(t'+\tau)Y(t')\right] \\ &= R_{X,Y}(\tau), \end{aligned}$$

Example

Example. Let $X(t)$ and $N(t)$ be two independent WSS random processes with expectations $\mathbb{E}[X(t)] = \mu_x$ and $\mathbb{E}[N(t)] = 0$, respectively. Let $Y(t) = X(t) + N(t)$. We want to show that $X(t)$ and $Y(t)$ are jointly WSS, and we want to find $R_{X,Y}(\tau)$.

Solution. Before we show the joint WSS property of $X(t)$ and $Y(t)$, we first show that $Y(t)$ is WSS:

$$
\begin{aligned}
\mathbb{E}[Y(t)] &= \mathbb{E}[X(t) + N(t)] = \mu_x \\
R_Y(t_1, t_2) &= \mathbb{E}\left[(X(t_1) + N(t_1))(X(t_2) + N(t_2))\right] \\
&= \mathbb{E}\left[(X(t_1)X(t_2)\right] + \mathbb{E}\left[(N(t_1)N(t_2)\right] \\
&= R_X(t_1 - t_2) + R_N(t_1 - t_2).
\end{aligned}
$$

Thus, $Y(t)$ is WSS.

To show that $X(t)$ and $Y(t)$ are jointly WSS, we need to check the cross-correlation function:

$$\begin{aligned} R_{X,Y}(t_1, t_2) &= \mathbb{E}[X(t_1)Y(t_2)] \\ &= \mathbb{E}\left[X(t_1)(X(t_2) + N(t_2))\right] \\ &= \mathbb{E}\left[X(t_1)(X(t_2)\right] + \mathbb{E}\left[X(t_1)N(t_2))\right] \\ &= R_X(t_1, t_2) + \mathbb{E}[X(t_1)]\mathbb{E}[N(t_2)] = R_X(t_1, t_2). \end{aligned}$$

Since $R_{X,Y}(t_1, t_2)$ is a function of $t_1 - t_2$, and since $X(t)$ and $Y(t)$ are WSS, $X(t)$ and $Y(t)$ must be jointly WSS.

Finally, to find $R_{X,Y}(\tau)$, we substitute $\tau = t_1 - t_2$ and obtain $R_{X,Y}(\tau) = R_X(\tau)$. □

Finding the cross-correlation

Theorem

Let $X(t)$ and $Y(t)$ be jointly WSS processes, and that $Y(t) = h(t) * X(t)$. Then the **cross-correlation** *$R_{Y,X}(\tau)$ is*

$$R_{Y,X}(\tau) = h(\tau) * R_X(\tau). \tag{2}$$

$$\begin{aligned} R_{Y,X}(\tau) &= \mathbb{E}[Y(t+\tau)X(t)] \\ &= \mathbb{E}\left[X(t)\int_{-\infty}^{\infty} X(t+\tau-r)h(r)dr\right] \\ &= \int_{-\infty}^{\infty} \mathbb{E}[X(t)X(t+\tau-r)]h(r)dr = \int_{-\infty}^{\infty} R_X(\tau-r)h(r)dr, \end{aligned}$$

which is the convolution $R_{Y,X}(\tau) = h(\tau) * R_X(\tau)$.

Cross Power Spectral Density

Definition

The **cross power spectral density** of two jointly WSS processes $X(t)$ and $Y(t)$ is defined as

$$S_{X,Y}(\omega) = \mathcal{F}[R_{X,Y}(\tau)]$$
$$S_{Y,X}(\omega) = \mathcal{F}[R_{Y,X}(\tau)]$$

Theorem

For two jointly WSS random processes $X(t)$ and $Y(t)$, the cross power spectral density satisfies the property that

$$S_{X,Y}(\omega) = \overline{S_{Y,X}(\omega)}, \tag{3}$$

where $\overline{(\cdot)}$ denotes the complex conjugate.

Theorem

If $X(t)$ passes through an LTI system to yield $Y(t)$, then the **cross power spectral density** *is*

$$S_{Y,X}(\omega) = H(\omega)S_X(\omega)$$
$$S_{X,Y}(\omega) = \overline{H(\omega)}S_X(\omega)$$

Example

Example 5. Let $X(t)$ be a WSS random process with

$$R_X(\tau) = e^{-\tau^2/2}, \quad H(\omega) = e^{-\omega^2/2}.$$

Find $S_{X,Y}(\omega)$, $R_{X,Y}(\tau)$, $S_Y(\omega)$ and $R_Y(\tau)$.

Solution. First, by Fourier transform table we know that

$$S_X(\omega) = \sqrt{2\pi} e^{-\omega^2/2}.$$

Since $H(\omega) = e^{-\omega^2/2}$, we have

$$S_{X,Y}(\omega) = \overline{H(\omega)} S_X(\omega) = \sqrt{2\pi} e^{-\omega^2}.$$

The cross-correlation function is

$$R_{X,Y}(\omega) = \mathcal{F}^{-1}\left[\sqrt{2\pi}e^{-\omega^2}\right] = \frac{1}{\sqrt{2}}e^{-\frac{\tau^2}{4}}.$$

The power spectral density of $Y(t)$ is

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = \sqrt{2\pi}e^{-\frac{3\omega^2}{2}}.$$

Therefore, the autocorrelation function of $Y(t)$ is

$$R_Y(\tau) = \mathcal{F}^{-1}\left[\sqrt{2\pi}e^{-\frac{3\omega^2}{2}}\right] = \frac{1}{\sqrt{3}}e^{-\tau^2/6}.$$

Questions?