ECE 302: Lecture A.8 Mean and Autocorrelation through LTI Systems

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Mean function

Theorem

If $X(t)$ passes through an LTI system to yield $Y(t)$, then the mean function of $Y(t)$ is

$$\mathbb{E}[Y(t)] = \mu_X \int_{-\infty}^{\infty} h(\tau) d\tau. \tag{1}$$

Proof

Suppose that $Y(t) = h(t) * X(t)$. Then,

$$\begin{aligned}
\mu_Y(t) = \mathbb{E}[Y(t)] &= \mathbb{E}\left[\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau\right] \\
&= \int_{-\infty}^{\infty} h(\tau)\mathbb{E}[X(t-\tau)]d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\mu_X d\tau \\
&= \mu_X \quad \int_{-\infty}^{\infty} h(\tau)d\tau,
\end{aligned}$$

where the last equality holds because $X(t)$ is WSS so that $\mathbb{E}[X(t-\tau)] = \mu_X$.

Example

Example 1. Consider a WSS random process $X(t)$ such that each sample is an i.i.d. Gaussian random variable with zero mean and unit variance. We send this process through an LTI system with impulse response $h(t)$, where

$$h(t) = \begin{cases} 10(1-|t|), & -1 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mu_Y(t)$.
Solution. $X(t)$ is i.i.d. Gaussian, and so

$$\mu_X(t) = \mathbb{E}[X(t)] = 0. \tag{2}$$

Hence,

$$\mu_Y(t) = \mathbb{E}[Y(t)] = \mu_X \int_{-\infty}^{\infty} h(\tau) d\tau = 0. \tag{3}$$

Example

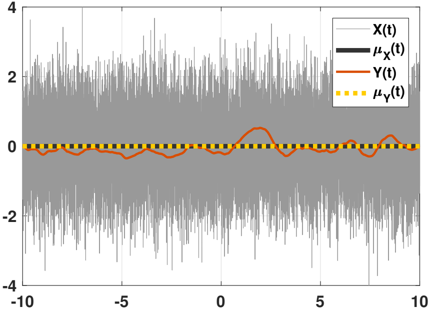

Figure: When sending a WSS random process through an LTI system, the mean and the autocorrelation functions are changed.

Autocorrelation

Theorem

If $X(t)$ passes through an LTI system to yield $Y(t)$, then the auto-correlation function of $Y(t)$ is

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)R_X(\tau + s - r)dsdr. \tag{4}$$

A short hand notation of the above formula is $R_Y(t) = [h \circledast (h * R_X)](t)$, where $*$ denotes the convolution and $\circledast$ denotes the correlation.

Proof

We start with the definition of $Y(t)$:

$$\begin{aligned} R_Y(\tau) &= \mathbb{E}[Y(t)Y(t+\tau)] \\ &= \mathbb{E}\left[\int_{-\infty}^{\infty} h(s)X(t-s)ds \int_{-\infty}^{\infty} h(r)X(t+\tau-r)dr\right] \\ &\overset{(a)}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)\mathbb{E}\left[X(t-s)X(t+\tau-r)dsdr\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)R_X(\tau+s-r)dsdr, \end{aligned}$$

where in (a) we assume that integration and expectation are interchangeable.

Example 2. Same as example 1: $X(t)$ is i.i.d. Gaussian, and

$$h(t) = \begin{cases} 10(1-|t|), & -1 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $R_Y(\tau)$.

Solution. $X(t)$ is i.i.d. Gaussian, and so

$$R_X(\tau) = \delta(\tau). \tag{5}$$

Hence,

$$\begin{aligned} R_Y(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)R_X(\tau+s-r)dsdr \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)\delta(\tau+s-r)dsdr \\ &= \int_{-\infty}^{\infty} h(s)h(\tau+s)ds \end{aligned}$$

Example

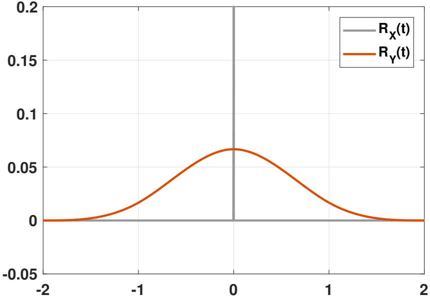

Figure: When sending a WSS random process through an LTI system, the mean and the autocorrelation functions are changed.

Power spectral density through LTI systems

Theorem

If $X(t)$ passes through an LTI system to yield $Y(t)$, then the power spectral density of $Y(t)$ is

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega). \tag{6}$$

Proof

By definition, power spectral density $S_Y(\omega)$ is the Fourier transform of the autocorrelation function $R_Y(\omega)$. Therefore,

$$
\begin{aligned}
S_Y(\omega) &= \int_{-\infty}^{\infty} R_Y(\tau) e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(s)h(r)R_X(\tau + s - r)dsdr \; e^{-j\omega\tau} d\tau
\end{aligned}
$$

Letting $u = \tau + s - r$, we have

$$
\begin{aligned}
S_Y(\omega) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(s)h(r)R_X(u)e^{-j\omega(u-s+r)} dsdrdu \\
&= \int_{-\infty}^{\infty} h(s)e^{j\omega s} ds \int_{-\infty}^{\infty} h(r)e^{-j\omega r} dr \int_{-\infty}^{\infty} R_X(u)e^{-j\omega u} du \\
&= \overline{H(\omega)} H(\omega) S_X(\omega),
\end{aligned}
$$

where $\overline{H(\omega)}$ is the complex conjugate of $H(\omega)$.

Example

Example 3: A WSS process $X(t)$ has a correlation function

$$R_X(\tau) = \mathrm{sinc}(\pi\tau).$$

Suppose that $X(t)$ passes through an LTI system with input/output relationship

$$2\frac{d^2}{dt^2}Y(t) + 2\frac{d}{dt}Y(t) + 4Y(t) = 3\frac{d^2}{dt^2}X(t) - 3\frac{d}{dt}X(t) + 6X(t).$$

Find $R_Y(\tau)$.

Solution: The sinc function has a Fourier transform given by

$$\text{sinc}(Wt) \underset{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{W}\text{rect}\left(\frac{\omega}{2W}\right).$$

Therefore, the auto-correlation function is

$$R_X(\tau) = \text{sinc}(\pi\tau) \quad \underset{\mathcal{F}}{\longleftrightarrow} \quad \frac{\pi}{\pi}\text{rect}\left(\frac{\omega}{2\pi}\right).$$

By taking the Fourier transform on both sides, we have

$$S_X(\omega) = \begin{cases} 1, & -\pi \le \omega \le \pi, \\ 0, & \text{elsewhere.} \end{cases}$$

The system response can be found from the differential equation as

$$H(\omega) = \frac{3(j\omega)^2 - 3(j\omega) + 6}{2(j\omega)^2 + 2(j\omega) + 4} = \frac{3\left[(2-\omega^2) - j\omega\right]}{2\left[(2-\omega^2) + j\omega\right]}.$$

Taking the magnitude square yields

$$|H(\omega)|^2 = \frac{3\left[(2-\omega^2) - j\omega\right]}{2\left[(2-\omega^2) + j\omega\right]} \frac{3\left[(2-\omega^2) + j\omega\right]}{2\left[(2-\omega^2) - j\omega\right]} = \frac{9}{4}\frac{(2-\omega^2)^2 + \omega^2}{(2-\omega^2)^2 + \omega^2} = \frac{9}{4}.$$

Therefore, the output power spectral density is

$$
\begin{aligned}
S_Y(\omega) &= |H(\omega)|^2 S_X(\omega) \\
&= \frac{9}{4} S_X(\omega).
\end{aligned}
$$

Taking the inverse Fourier transform, we have

$$
R_Y(\tau) = \frac{9}{4} \mathrm{sinc}(\pi\tau).
$$

Questions?