ECE 302: Lecture A.3 Autocorrelation Function

Prof Stanley Chan

School of Electrical and Computer Engineering
Purdue University
What is a random process?

A random process is a function indexed by a random key.
The **autocorrelation function** of a random process $X(t)$ is

$$R_X(t_1, t_2) = \mathbb{E} [X(t_1)X(t_2)].$$

Autocorrelation function takes two time instants $t_1$ and $t_2$. Since $X(t_1)$ and $X(t_2)$ are two random variables, $R_X(t_1, t_2) = \mathbb{E} [X(t_1)X(t_2)]$ measures the correlation of these two random variables.
Example 1: Let $A \sim \text{Uniform}[0, 1]$, $X(t) = A \cos(2\pi t)$. Find $R_X(t_1, t_2)$.

Solution.

$$R_X(t_1, t_2) = \mathbb{E}[A \cos(2\pi t_1)A \cos(2\pi t_2)]$$

$$= \mathbb{E}[A^2] \cos(2\pi t_1) \cos(2\pi t_2)$$

$$= \frac{1}{3} \cos(2\pi t_1) \cos(2\pi t_2).$$
Example

**Example 2:** Let $\Theta \sim \text{Uniform}[-\pi, \pi]$, $X(t) = \cos(\omega t + \Theta)$. Find $R_X(t_1, t_2)$.

**Solution.**

$$R_X(t_1, t_2) = \mathbb{E} [\cos(\omega t_1 + \Theta) \cos(\omega t_2 + \Theta)]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t_1 + \theta) \cos(\omega t_2 + \theta) d\theta$$

$$\overset{(a)}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \left[ \cos(\omega(t_1 + t_2) + 2\theta) + \cos(\omega(t_1 - t_2)) \right] d\theta,$$

$$= \frac{1}{2} \cos \left( \omega(t_1 - t_2) \right),$$

where in (a) we applied the trigonometric formula $\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)]$. 
Understanding auto-correlation function

How to understand the meaning of $E[X(t_1)X(t_2)]$?

$E[X(t_1)X(t_2)]$ is analogous to the correlation $E[XY]$. 
Example

Example 3. Let \( X(t) = A \cos(2\pi t) \). Find \( \mathbb{E}[X(0)X(0.5)] \).

Solution. If \( X(t) = A \cos(2\pi t) \), then

\[
X(0) = A \cos(0) = A, \\
X(0.5) = A \cos(\pi) = -A.
\]

When you have two random variables, you can talk about their correlations. Using this example, we can show that

\[
\mathbb{E}[X(0)X(0.5)] = -\mathbb{E}[A \cdot A] = -\mathbb{E}[A^2] = -\frac{1}{3}.
\]
Example

You have two PDFs:

\[ f_{X(0)}(x) = \begin{cases} 
1, & 0 \leq x \leq 1, \\
0, & \text{otherwise}
\end{cases} \quad \text{and} \quad f_{X(0.5)}(x) = \begin{cases} 
1, & -1 \leq x \leq 0, \\
0, & \text{otherwise}
\end{cases} \]

**Figure:** The autocorrelation between \( X(0) \) and \( X(0.5) \) should be regarded as the correlation between two random variables. Each random variable has its states, and its probabilities.
Example 4: Let $A \sim \text{Uniform}[0, 1]$, $X(t) = A \cos(2\pi t)$. Find $R_X(0, 0.5)$, and draw $R_X(t_1, t_2)$.

Solution. In the previous example, we know that

$$R_X(t_1, t_2) = \frac{1}{3} \cos(2\pi t_1) \cos(2\pi t_2).$$

Therefore, $R_X(0, 0.5) = \frac{1}{3} \cos(2\pi 0) \cos(2\pi 0.5) = -\frac{1}{3}$ which is the same as if we compute it from the first principle.
Visualizing the autocorrelation function

Figure: The autocorrelation function \( R_X(t_1, t_2) = \frac{1}{3} \cos(2\pi t_1) \cos(2\pi t_2) \).
Example

**Example 5:** Let $\Theta \sim \text{Uniform}[-\pi, \pi]$, $X(t) = \cos(\omega t + \Theta)$. Draw the auto-correlation function $R_X(t_1, t_2)$.

**Solution.** From the previous example we know that

$$R_X(t_1, t_2) = \frac{1}{2} \cos \left( \omega (t_1 - t_2) \right).$$

**Figure:** The autocorrelation function $R_X(t_1, t_2) = \frac{1}{2} \cos \left( \omega (t_1 - t_2) \right)$. 

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**Note:** The diagrams illustrate the behavior of $R_X(t_1, t_2)$ for different values of $t_1$ and $t_2$. The left diagram shows the waveforms of $X(t)$ for various values of $\Theta$, while the right diagram visualizes the correlation matrix $R_X(t_1, t_2)$ with $t_1$ and $t_2$ on the axes.
The **autocorrelation function** of a random process \( X(t) \) is

\[
R_X(t_1, t_2) = \mathbb{E} [X(t_1)X(t_2)] .
\]

- \( R_X(t_1, t_2) \) is a 2D function
- It measures the correlation between two *random variables* \( X(t_1) \) and \( X(t_2) \)
- Same meaning as \( \mathbb{E}[XY] \)
Questions?