ECE 302: Lecture A.3 Autocorrelation Function

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Everything you need to know about a random process

What is a random process?

A random process is a function indexed by a random key.

Autocorrelation function

The **autocorrelation function** of a random process $X(t)$ is

$$R_X(t_1, t_2) = \mathbb{E}\left[X(t_1)X(t_2)\right].$$

Autocorrelation function takes two time instants t_1 and t_2. Since $X(t_1)$ and $X(t_2)$ are two random variables, $R_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)]$ measures the correlation of these two random variables.

Example

Example 1: Let $A \sim$Uniform$[0, 1]$, $X(t) = A\cos(2\pi t)$. Find $R_X(t_1, t_2)$.

Solution.

$$\begin{aligned} R_X(t_1, t_2) &= \mathbb{E}\left[A\cos(2\pi t_1)A\cos(2\pi t_2)\right] \\ &= \mathbb{E}[A^2]\cos(2\pi t_1)\cos(2\pi t_2) \\ &= \frac{1}{3}\cos(2\pi t_1)\cos(2\pi t_2). \end{aligned}$$

Example

Example 2: Let $\Theta \sim$Uniform$[-\pi, \pi]$, $X(t) = \cos(\omega t + \Theta)$. Find $R_X(t_1, t_2)$.

Solution.

$$\begin{aligned}
R_X(t_1, t_2) &= \mathbb{E}\left[\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta) d\theta \\
&\overset{(a)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\Big[\cos(\omega(t_1 + t_2) + 2\theta) + \cos(\omega(t_1 - t_2))\Big] d\theta, \\
&= \frac{1}{2}\cos\Big(\omega(t_1 - t_2)\Big),
\end{aligned}$$

where in (a) we applied the trigonometric formula $\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$.

Understanding auto-correlation function

How to understand the meaning of $\mathbb{E}[X(t_1)X(t_2)]$?

$$\mathbb{E}\left[X(t_1)X(t_2)\right] \text{ is analogous to the correlation } \mathbb{E}[XY]$$

Example

Example 3. Let $X(t) = A\cos(2\pi t)$. Find $\mathbb{E}[X(0)X(0.5)]$.

Solution. If $X(t) = A\cos(2\pi t)$, then

$$X(0) = A\cos(0) = A,$$
$$X(0.5) = A\cos(\pi) = -A.$$

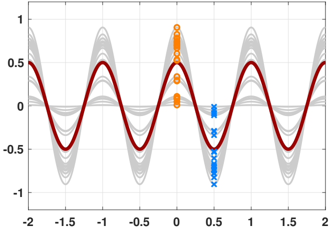
When you have two random variables, you can talk about their correlations. Using this example, we can show that

$$\mathbb{E}[X(0)X(0.5)] = -\mathbb{E}[A \cdot A] = -\mathbb{E}[A^2] = -\frac{1}{3}.$$

Example

You have two PDFs:

$$f_{X(0)}(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad f_{X(0.5)}(x) = \begin{cases} 1, & -1 \le x \le 0, \\ 0, & \text{otherwise} \end{cases}$$



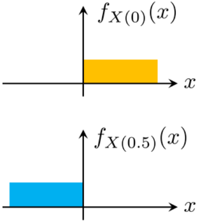


Figure: The autocorrelation between $X(0)$ and $X(0.5)$ should be regarded as the correlation between two random variables. Each random variable has its states, and its probabilities.

Example

Example 4: Let $A \sim$Uniform$[0,1]$, $X(t) = A\cos(2\pi t)$. Find $R_X(0, 0.5)$, and draw $R_X(t_1, t_2)$.

Solution. In the previous example, we know that

$$R_X(t_1, t_2) = \frac{1}{3}\cos(2\pi t_1)\cos(2\pi t_2).$$

Therefore, $R_X(0, 0.5) = \frac{1}{3}\cos(2\pi 0)\cos(2\pi 0.5) = -\frac{1}{3}$ which is the same as if we compute it from the first principle.

Visualizing the autocorrelation function

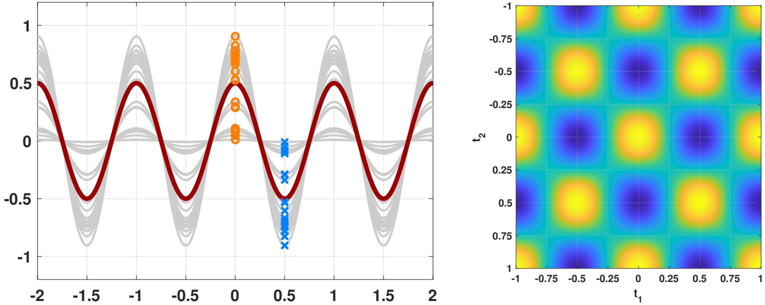


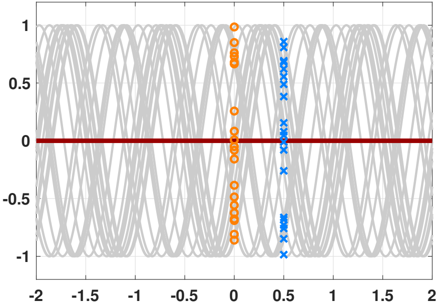
Figure: The autocorrelation function $R_X(t_1, t_2) = \frac{1}{3}\cos(2\pi t_1)\cos(2\pi t_2)$.

Example

Example 5: Let $\Theta \sim$Uniform$[-\pi, \pi]$, $X(t) = \cos(\omega t + \Theta)$. Draw the auto-correlation function $R_X(t_1, t_2)$.

Solution. From the previous example we know that

$$R_X(t_1, t_2) = \frac{1}{2}\cos\Big(\omega(t_1 - t_2)\Big).$$



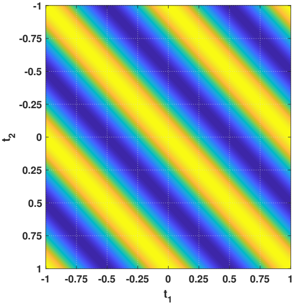


Figure: The autocorrelation function $R_X(t_1, t_2) = \frac{1}{2}\cos\Big(\omega(t_1 - t_2)\Big)$.

Summary

The **autocorrelation function** of a random process $X(t)$ is

$$R_X(t_1, t_2) = \mathbb{E}\left[X(t_1)X(t_2)\right].$$

- $R_X(t_1, t_2)$ is a 2D function
- It measures the correlation between two *random variables* $X(t_1)$ and $X(t_2)$
- Same meaning as $\mathbb{E}[XY]$

Questions?