ECE 302: Lecture A.1 Random Process

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Everything you need to know about a random process

What is a random process?

A random process is a function indexed by a random key.

Example 1

Example 1. We define two random variables a and b, which are uniformly distributed in certain range. We then define a function:

$$f(t) = at + b, \qquad -2 \leq t \leq 2. \tag{1}$$

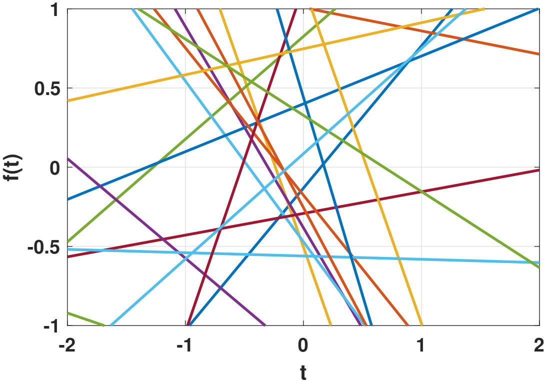

Figure: The set of straight lines $A = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = ax + b,\ a, b \in \mathbb{R}\}$.

Example 1

So, to be more formal:

$$f(t,\xi) = a(\xi)t + b(\xi), \qquad -2 \le t \le 2, \quad \xi \in \Omega,$$

Example:

- Suppose samples space is $(a, b) = (1.2, 0.6)$ and $(a, b) = (-0.75, 1.8)$.
- Probability of getting either pair is $\frac{1}{2}$
- Then,

$$f(t,\xi) = \begin{cases} 1.2t + 0.6, & \text{with probability } \frac{1}{2}, \\ -0.75t + 1.8, & \text{with probability } \frac{1}{2}. \end{cases}$$

Example 2

Example 2. Consider the function

$$f(t) = \cos(\omega_0 t + \Theta), \qquad -1 \leq t \leq 1,$$

where Θ is a random phase distributed uniformly over the range $[0, 2\pi]$.

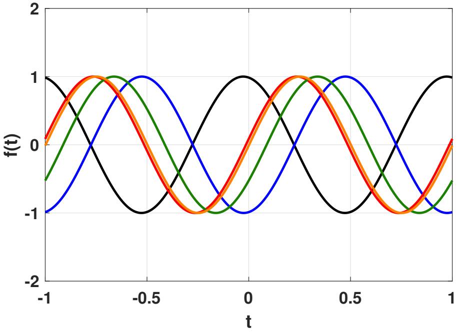

Figure: The set of phase shifted cosines
$A = \{f : \mathbb{R} \to [-1, 1] \mid f(t) = \cos(\omega_0 t + \theta),\ \theta \in [0, 2\pi]\}$.

Example 2

To be more formal:

$$f(t, \xi) = \cos(\omega_0 t + \Theta(\xi)), \qquad -1 \leq t \leq 1, \quad \xi \in \Omega. \tag{2}$$

Again, ξ denotes the index of the random variable Θ. Since Θ is drawn uniformly from the interval $[0, 2\pi]$, the following functions are possible realizations:

$$f(t, \xi_1) = \cos\left(\omega_0 t + \frac{3\pi}{4}\right), \qquad -1 \leq t \leq 1,$$
$$f(t, \xi_2) = \cos\left(\omega_0 t - \frac{7\pi}{3}\right), \qquad -1 \leq t \leq 1.$$

Sample space of random processes

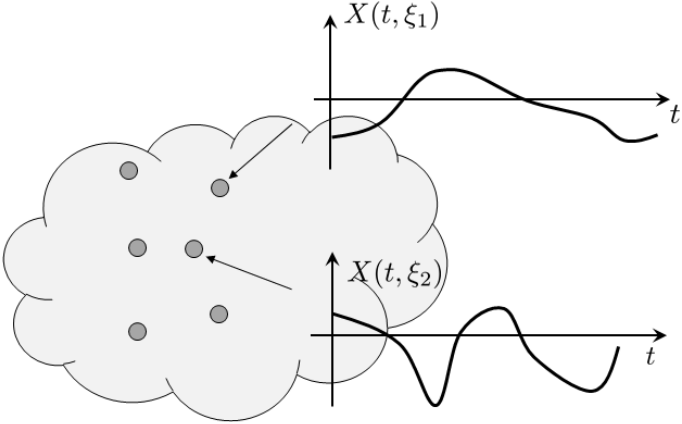

Figure: The sample space of a random process $X(t,\xi)$ contains many functions. Therefore, each random realization is a function.

Statistical and temporal view of a random process

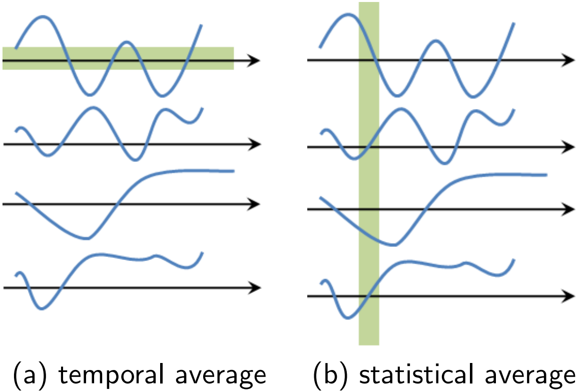

Figure: Temporal and statistical perspectives of a random process.

Statistical and temporal view

Statistical View: Fix time t. We can look at the 2-dimensional function $X(t, \xi)$ "vertically" as

$$\begin{cases} X(t, \xi_1) \\ X(t, \xi_2) \\ \vdots \\ X(t, \xi_N) \end{cases}$$

This is a sequence of **random variables** because $\xi_1, \ldots, \xi_N$ are realizations of the random variable ξ.

Temporal View: Fix the random index ξ. We can look at $X(t, \xi)$ "horizontally" as

$$X(t_1, \xi), X(t_2, \xi), \ldots, X(t_K, \xi).$$

This is a **deterministic** time series evaluated at time points $t_1, \ldots, t_K$.

Example 3

Random Process. Let $A \sim$Uniform$[0,1]$. Define $X(t,\xi) = A(\xi)\cos(2\pi t)$.

- **Statistical View**: Fix t (for example $t = 10$). In this case, we have

$$X(t,\xi) = A(\xi)\cos(2\pi(10)) = A(\xi)\cos(20\pi),$$

  which is a random variable because $\cos(20\pi)$ is a constant. The randomness of X comes from the fact that $A(\xi) \sim$ Uniform$[0,1]$.

- **Temporal View**: Fix ξ (for example $A(\xi) = 0.7$). In this case, we have

$$X(t,\xi) = 0.7\cos(2\pi t),$$

  which is a deterministic function in t.

Example 3

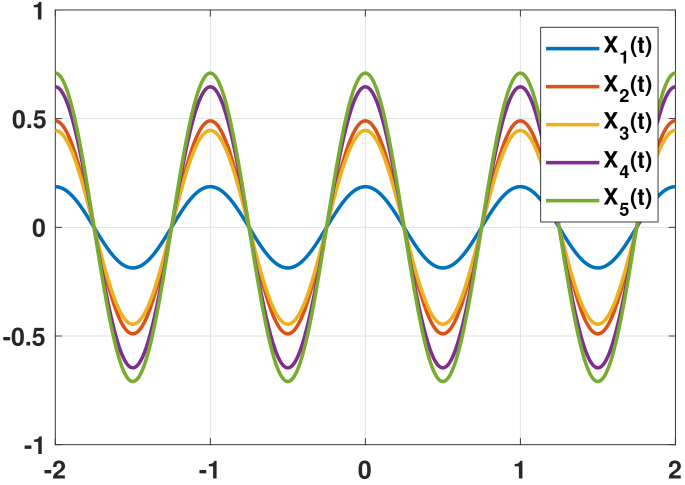

Figure: Realizations of the random process $X(t) = A\cos(2\pi t)$.

Example 4

Random Process. Let A be a discrete random variable with PMF

$$\mathbb{P}(A=+1)=\frac{1}{2}, \text{ and } \mathbb{P}(A=-1)=\frac{1}{2}.$$

Define $X(n,\xi)=A(\xi)(-1)^n$.

- **Statistical View**: Fix n, say $n=10$. Then,

$$X(n,\xi)=X(10,\xi)=\begin{cases}(-1)^{10}=1, & \text{with prob } 1/2\\ (-1)^{11}=-1, & \text{with prob } 1/2,\end{cases}$$

  which is a random variable.

- **Temporal View**: Fix ξ. Then,

$$X(n,\xi)=\begin{cases}(-1)^{n}, & \text{if } A=+1\\ (-1)^{n+1}, & \text{if } A=-1,\end{cases}$$

  which is a time series.

Example 4

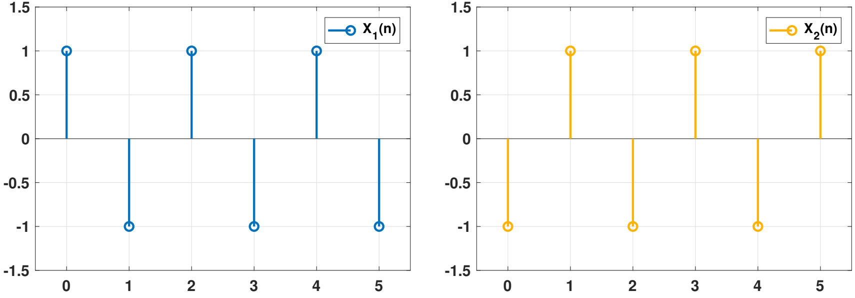

Figure: Realizations of the random process $X[n] = A(-1)^n$.

Summary

What do we mean by statistical average and temporal average?

- Statistical average: Take average of $X(t, \xi)$ over ξ. This is the vertical average.
- Temporal average: Take average of $X(t, \xi)$ over t. This is the horizontal average.
- In general, statistical average $\neq$ temporal average.

Questions?