ECE 302: Lecture 6.2 Characteristic Function

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Characteristic Function

Definition (This course)

The **characteristic function** of a random variable X is

$$\Phi_X(j\omega) = \mathbb{E}[e^{-j\omega X}]. \tag{1}$$

Definition (Common definition)

The **characteristic function** of a random variable X is

$$\Phi_X(j\omega) = \mathbb{E}[e^{j\omega X}]. \tag{2}$$

Recall MGF is

$$M_X(s) = \mathbb{E}\left[e^{sX}\right]. \tag{3}$$

So, $\Phi_X(j\omega)$ is MGF evaluated at $s = -j\omega$.

What is characteristic function then?

Definition (This course)

The **characteristic function** of a random variable X is

$$\Phi_X(j\omega) = \mathbb{E}[e^{-j\omega X}]. \tag{4}$$

Characteristic function is the Fourier transform of $f_X(x)$:

$$\Phi_X(j\omega) = \mathbb{E}[e^{-j\omega X}] = \int_{-\infty}^{\infty} e^{-j\omega x} f_X(x) dx. \tag{5}$$

Fourier transform Table

	$f(t) \longleftrightarrow F(\omega)$		$f(t) \longleftrightarrow F(\omega)$
1.	$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$, $a>0$	10.	$\mathrm{sinc}^2(\frac{Wt}{2}) \longleftrightarrow \frac{2\pi}{W}\Delta(\frac{\omega}{2W})$
2.	$e^{at}u(-t) \longleftrightarrow \frac{1}{a-j\omega}$, $a>0$	11.	$e^{-at}\sin(\omega_0 t)u(t) \longleftrightarrow \frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$
3.	$e^{-a\lvert t\rvert} \longleftrightarrow \frac{2a}{a^2+\omega^2}$, $a>0$	12.	$e^{-at}\cos(\omega_0 t)u(t) \longleftrightarrow \frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$
4.	$\frac{a^2}{a^2+t^2} \longleftrightarrow \pi a e^{-a\lvert\omega\rvert}$, $a>0$	13.	$e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sqrt{2\pi}\sigma e^{-\frac{\sigma^2\omega^2}{2}}$
5.	$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+j\omega)^2}$, $a>0$	14.	$\delta(t) \longleftrightarrow 1$
6.	$t^n e^{-at}u(t) \longleftrightarrow \frac{n!}{(a+j\omega)^{n+1}}$, $a>0$	15.	$1 \longleftrightarrow 2\pi\delta(\omega)$
7.	$\mathrm{rect}(\frac{t}{\tau}) \longleftrightarrow \tau\,\mathrm{sinc}(\frac{\omega\tau}{2})$	16.	$\delta(t-t_0) \longleftrightarrow e^{-j\omega t_0}$
8.	$\mathrm{sinc}(Wt) \longleftrightarrow \frac{\pi}{W}\mathrm{rect}(\frac{w}{2W})$	17.	$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0)$
9.	$\Delta(\frac{t}{\tau}) \longleftrightarrow \frac{\tau}{2}\mathrm{sinc}^2(\frac{\omega\tau}{4})$	18.	$f(t)e^{j\omega_0 t} \longleftrightarrow F(\omega-\omega_0)$

Example 1

Example. Let X be a random variable with PDF $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Find the characteristic function.

Solution. The Fourier transform pair is

$$\lambda e^{-\lambda x} \longrightarrow \lambda \cdot \mathcal{F}\left\{ e^{-\lambda x} \right\} = \lambda \cdot \frac{1}{\lambda + j\omega}.$$

Therefore, the characteristic function is $\Phi_X(j\omega) = \frac{\lambda}{\lambda + j\omega}$.

Example 2

Example. Let X and Y be independent, and let

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y \geq 0 \\ 0, & y < 0 \end{cases}.$$

Find the PDF of $Z = X + Y$.

Example 2

Solution. The characteristic function of X and Y can be found from the Fourier table:

$$\Phi_X(j\omega) = \frac{\lambda}{\lambda + j\omega}, \qquad \text{and} \qquad \Phi_Y(j\omega) = \frac{\lambda}{\lambda + j\omega}.$$

Therefore, the characteristic function of Z is

$$\Phi_Z(j\omega) = \Phi_X(j\omega)\Phi_Y(j\omega) = \frac{\lambda^2}{(\lambda + j\omega)^2}.$$

By inverse Fourier transform, we have that

$$f_Z(z) = \mathcal{F}^{-1}\left\{\frac{\lambda^2}{(\lambda + j\omega)^2}\right\} = \lambda^2 z e^{-\lambda z}, \quad z \geq 0.$$

Example 3

Example. Let $X_0, X_1, \ldots$ be a sequence of independent random variables with PDF

$$f_{X_k}(x) = \frac{a_k}{\pi(a_k^2 + x^2)}, \qquad a_k = \frac{1}{2^{k+1}},$$

for $k = 0, 1, \ldots$. Find the PDF of Y, where $Y = \sum_{k=0}^{\infty} X_k$.

Example 3

Solution. Using the Fourier transform table, we know that

$$\frac{a_k}{\pi(a_k^2 + x^2)} \overset{\mathcal{F}}{\longleftrightarrow} \qquad .$$

The characteristic function of Y is

$$\Phi_Y(j\omega) = \qquad = \exp\left\{-|\omega| \sum_{k=0}^{\infty} a_k\right\}.$$

Since $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2} + \frac{1}{4} + \ldots = 1$, the characteristic function becomes $\Phi_Y(j\omega) = e^{-|\omega|}$. Inverse Fourier transform will give us

$$e^{-|\omega|} \overset{\mathcal{F}}{\longleftrightarrow} \qquad .$$

Therefore the PDF of Y is

$$f_Y(y) = \frac{1}{\pi(1 + y^2)}. \tag{6}$$

Why $\Phi_X(j\omega)$ but not $M_X(s)$?

- MGF does not always exists

Theorem

Consider the Cauchy distribution with PDF

$$f_X(x) = \frac{1}{\pi(x^2+1)}.$$

The MGF of X is undefined, but the characteristic function is well defined.

Proof

Proof. It can be shown that the MGF is

$$\begin{aligned} M_X(s) &= \int_{-\infty}^{\infty} e^{sx} \frac{1}{\pi(x^2+1)} dx \geq \int_{1}^{\infty} e^{sx} \frac{1}{\pi(x^2+1)} dx \\ &\geq \int_{1}^{\infty} \frac{(sx)^3}{6\pi(x^2+1)} dx, \quad \text{because} \quad e^{sx} \geq \frac{(sx)^3}{6} \\ &\geq \int_{1}^{\infty} \frac{(sx)^3}{6\pi(2x^2)} dx = \frac{s^3}{12\pi} \int_{1}^{\infty} x dx = \infty. \end{aligned}$$

Therefore, the MGF is undefined. On the other hand, by Fourier table we know that

$$\Phi_X(j\omega) = \mathcal{F}\left\{ \frac{1}{\pi(x^2+1)} \right\} = e^{-|\omega|}. \qquad \square$$

Summary

What is a characteristic function?

- $\mathbb{E}[e^{-j\omega X}]$
- Also okay to define as $\mathbb{E}[e^{j\omega X}]$
- Fourier transform of PDF
- Useful for sum of random variables

Questions?