ECE 302: Lecture 5.8 Random Vectors

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More than two random variables?

Joint distributions are **high-dimensional** PDF (or PMF or CDF).

$$\underbrace{f_X(x)}_{\text{one variable}} \Longrightarrow \underbrace{f_{X_1,X_2}(x_1,x_2)}_{\text{two variables}} \Longrightarrow \underbrace{f_{X_1,X_2,X_3}(x_1,x_2,x_3)}_{\text{three variables}}$$

$$\Longrightarrow \ldots \Longrightarrow \underbrace{f_{X_1,\ldots,X_N}(x_1,\ldots,x_N)}_{N \text{ variables}}.$$

Notation:

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{X_1,\ldots,X_N}(x_1,\ldots,x_N).$$

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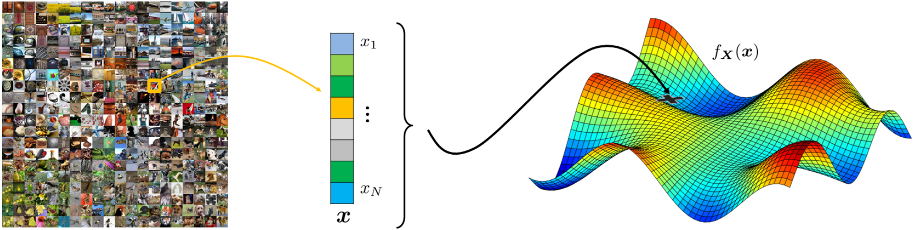
Random vectors in practice



- Joint distributions are ubiquitous in modern data analysis.
- For example, an image from a dataset can be represented by a high-dimensional vector $\boldsymbol{x}$.
- Each vector has certain probability to be present.
- Such probability is described by the high-dimensional joint PDF $f_{\boldsymbol{X}}(\boldsymbol{x})$.

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Random Vectors

Random vector:

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}, \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}.$$

Joint PDF:

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{X_1,X_2,\ldots,X_N}(x_1, x_2, \ldots, x_N). \tag{1}$$

Probability:

$$\begin{aligned} \mathbb{P}[\boldsymbol{X} \in \mathcal{A}] &= \int_{\mathcal{A}} f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \\ &= \int \cdots \int_{\mathcal{A}} f_{X_1,\ldots,X_N}(x_1, \ldots, x_N) dx_1 \ldots dx_N. \end{aligned}$$

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Independence

If the elements are independent, then

$$f_{X_1,\ldots,X_N}(x_1,\ldots,x_N) = f_{X_1}(x_1) f_{X_2}(x_2) \ldots f_{X_N}(x_N),$$

Example. Let $\boldsymbol{X} = [X_1,\ldots,X_N]^T$ be a vector of zero-mean unit variance Gaussian random vectors. Let $\mathcal{A} = [-1,2]^N$. Then,

$$\begin{aligned}
\mathbb{P}[\boldsymbol{X} \in \mathcal{A}] &= \int_{\mathcal{A}} f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \\
&= \int \ldots \int_{\mathcal{A}} f_{X_1,\ldots,X_N}(x_1,\ldots,x_N) dx_1 \ldots dx_N \\
&= \left[\int_{-1}^{2} f_{X_1}(x_1) dx_1 \right]^N = [\Phi(2) - \Phi(-1)]^N,
\end{aligned}$$

where $\Phi(\cdot)$ is the standard Gaussian CDF.

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Mean vector

Definition

Let $\boldsymbol{X} = [X_1, \ldots, X_N]^T$ be a random vector. The expectation is

$$\boldsymbol{\mu} \overset{\text{def}}{=} \mathbb{E}[\boldsymbol{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \\ \vdots \\ \mathbb{E}[X_N] \end{bmatrix} \tag{2}$$

How to compute the mean vector:

$$\mathbb{E}[\boldsymbol{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \vdots \\ \mathbb{E}[X_N] \end{bmatrix} = \begin{bmatrix} \int_\Omega x_1 f_{X_1}(x_1) dx_1 \\ \vdots \\ \int_\Omega x_N f_{X_N}(x_N) dx_N, \end{bmatrix}$$

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Example

Example. Let $\boldsymbol{X} = [X_1, \ldots, X_N]^T$ be a random vector such that X_n are independent Poissons with $X_n \sim \text{Poisson}(\lambda_n)$. Then

$$\mathbb{E}[\boldsymbol{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \vdots \\ \mathbb{E}[X_N] \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} k \cdot \frac{\lambda_1^k e^{-\lambda_1}}{k!} \\ \vdots \\ \sum_{k=0}^{\infty} k \cdot \frac{\lambda_N^k e^{-\lambda_N}}{k!} \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix}.$$

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Covariance matrix

Definition

Let $\boldsymbol{X} = [X_1, \ldots, X_N]^T$ be a random vector. The **covariance matrix** is

$$\boldsymbol{\Sigma} \stackrel{\text{def}}{=} \mathrm{Cov}(\boldsymbol{X}) = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}(X_1, X_2) & \ldots & \mathrm{Cov}(X_1, X_N) \\ \mathrm{Cov}[X_2, X_1] & \mathrm{Var}[X_2] & \ldots & \mathrm{Cov}(X_2, X_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(X_N, X_1) & \mathrm{Cov}(X_N, X_2) & \ldots & \mathrm{Var}[X_N]. \end{bmatrix} \tag{3}$$

A more compact way of writing the covariance matrix is

$$\boldsymbol{\Sigma} = \mathrm{Cov}(\boldsymbol{X}) = \mathbb{E}[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T], \tag{4}$$

where $\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{X}]$ is the mean vector.

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Diagonal covariance matrix

Theorem

If the coordinates $X_1, \ldots, X_N$ are independent, then the covariance matrix $\mathrm{Cov}(\boldsymbol{X}) = \boldsymbol{\Sigma}$ is a diagonal matrix:

$$\boldsymbol{\Sigma} = \mathrm{Cov}(\boldsymbol{X}) = \begin{bmatrix} \mathrm{Var}[X_1] & 0 & \ldots & 0 \\ 0 & \mathrm{Var}[X_2] & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathrm{Var}[X_N] \end{bmatrix}.$$

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Correlation matrix

Definition

Let $\boldsymbol{X} = [X_1, \ldots, X_N]^T$ be a random vector. The auto-correlation matrix is

$$\boldsymbol{R} = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^T] = \begin{bmatrix} \mathbb{E}[X_1 X_1] & \mathbb{E}[X_1 X_2] & \ldots & \mathbb{E}[X_1 X_N] \\ \mathbb{E}[X_2 X_1] & \mathbb{E}[X_2 X_2] & \ldots & \mathbb{E}[X_2 X_N] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[X_N X_1] & \mathbb{E}[X_N X_2] & \ldots & \mathbb{E}[X_N X_N]. \end{bmatrix} \tag{5}$$

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Multi-variate Gaussian

Definition

A d-dimensional **joint Gaussian** has a PDF

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^d |\boldsymbol{\Sigma}|}} \exp\left\{ -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) \right\}, \tag{6}$$

where d denotes the dimensionality of the vector $\boldsymbol{x}$.

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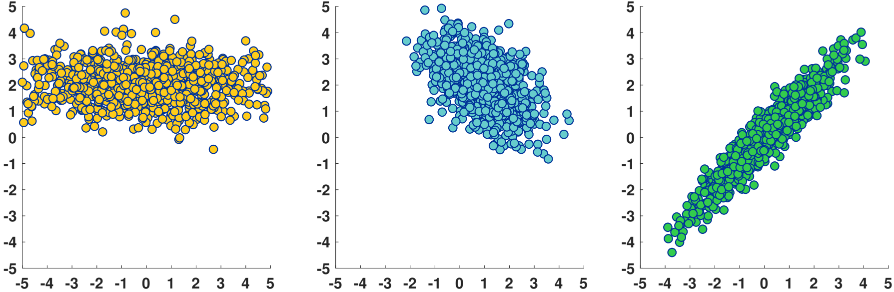
Multi-variate Gaussian



$$(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 0.5 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1.9 \\ 1.9 & 2 \end{bmatrix}$$

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Summary

Random vector:

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}, \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}.$$

Mean vector:

$$\boldsymbol{\mu} \overset{\text{def}}{=} \mathbb{E}[\boldsymbol{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \\ \vdots \\ \mathbb{E}[X_N] \end{bmatrix} \tag{7}$$

Covariance:

$$\boldsymbol{\Sigma} \overset{\text{def}}{=} \text{Cov}(\boldsymbol{X}) = \begin{bmatrix} \text{Var}[X_1] & \text{Cov}(X_1, X_2) & \dots & \text{Cov}(X_1, X_N) \\ \text{Cov}[X_2, X_1] & \text{Var}[X_2] & \dots & \text{Cov}(X_2, X_N) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(X_N, X_1) & \text{Cov}(X_N, X_2) & \dots & \text{Var}[X_N] \end{bmatrix} \tag{8}$$

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Questions?

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