

ECE 302: Lecture 5.5 Conditional Expectation

Prof Stanley Chan

School of Electrical and Computer Engineering
Purdue University



Conditional PMF and PDF

Definition

Let X and Y be two discrete random variables. The **conditional PMF** of X given Y is

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}. \quad (1)$$

Definition

Let X and Y be two continuous random variables. The **conditional PDF** of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}. \quad (2)$$

Outline

- Joint PDF and CDF
- Joint Expectation
- Conditional Distribution
- Conditional Expectation
- Sum of Two Random Variables
- Random Vectors
- High-dimensional Gaussians and Transformation
- Principal Component Analysis

Today's lecture

- What is conditional expectation
- Law of total expectation
- Examples

Conditional Expectation

Definition

The conditional expectation of X given $Y = y$ is

$$\mathbb{E}[X | Y = y] = \sum_x xp_{X|Y}(x|y) \quad (3)$$

for the discrete random variables, and

$$\mathbb{E}[X | Y = y] = \int_{-\infty}^{\infty} xf_{X|Y}(x|y)dx \quad (4)$$

for the continuous random variables.

What is conditional expectation?

What is conditional expectation?

- $\mathbb{E}[X|Y = y]$ is the expectation using $f_{X|Y}(x|y)$.
- The integration is taken w.r.t. x , because $Y = y$ is given and fixed.

Law of Total Expectation

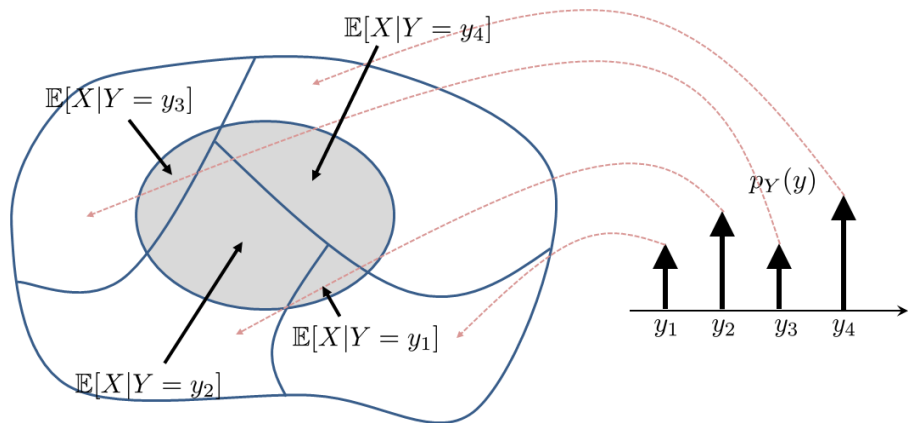
Theorem (Law of Total Expectation)

$$\mathbb{E}[X] = \sum_y \mathbb{E}[X|Y = y]p_Y(y), \quad \text{or} \quad \mathbb{E}[X] = \int_{-\infty}^{\infty} \mathbb{E}[X|Y = y]f_Y(y)dy. \quad (5)$$

What is law of total expectation?

- Law of total expectation is a **decomposition** rule.
- It decomposes $\mathbb{E}[X]$ into smaller/easier conditional expectations.

Law of Total Expectation



Proof

$$\begin{aligned}\mathbb{E}[X] &= \sum_x xp_X(x) \\ &= \\ &= \sum_x \sum_y xp_{X|Y}(x|y)p_Y(y) \\ &= \\ &= \sum_y \mathbb{E}[X|Y = y]p_Y(y). \quad \square\end{aligned}$$

$\mathbb{E}_{X|Y}[X|Y]$

$\mathbb{E}_{X|Y}[X|Y = y]$ = a deterministic function in y .

$\mathbb{E}_{X|Y}[X|Y]$ = a function of the random variable Y .

An alternative form

Some people prefer to write the above theorem in a more compact form.

Corollary

Let X and Y be two random variables. Then,

$$\mathbb{E}[X] = \mathbb{E}_Y [\mathbb{E}_{X|Y}[X|Y]]. \quad (6)$$

Proof.

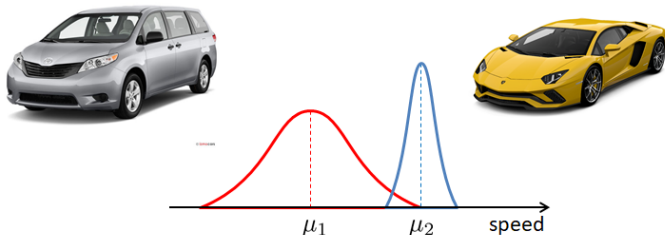
$$\begin{aligned} \mathbb{E}[X] &= \sum_y \mathbb{E}[X|Y = y]p_Y(y) \\ &= \sum_y h(y)p_Y(y) \\ &= \mathbb{E}_Y[h(Y)] = \mathbb{E}_Y [\mathbb{E}_{X|Y}[X|Y]]. \quad \square \end{aligned}$$

Example 1

Example 1. Suppose there are two classes of cars. Let X be the speed and C be the class.

- When $C = 1$, we know that $X \sim \text{Gaussian}(\mu_1, \sigma_1)$. We know that $\mathbb{P}[C = 1] = p$.
- When $C = 2$, $X \sim \text{Gaussian}(\mu_2, \sigma_2)$.
- Also, $\mathbb{P}[C = 2] = 1 - p$.

Suppose you see a car on the freeway, what is its average speed?



Example 1

Solution. The problem has given us everything we need. In particular, we know that the conditional PDFs are:

$$f_{X|C}(x | 1) =$$

$$f_{X|C}(x | 2) =$$

Therefore, conditioned on C , we have two expectations:

$$\mathbb{E}[X | C = 1] = \mu_1,$$

$$\mathbb{E}[X | C = 2] = \mu_2.$$

The overall expectation $\mathbb{E}[X]$ is

$$\mathbb{E}[X] =$$

Example 2

Example 2. Consider a joint PMF given by the following table.

Y	10^4	0	0	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$
	10^2	$\frac{5}{12}$	$\frac{5}{18}$	$\frac{5}{36}$	0	0
		0.01	0.1	1	10	100
				X		

Find $\mathbb{E}[X|Y = 10^2]$ and $\mathbb{E}[X|Y = 10^4]$.

Example 2

Solution. To find the conditional expectation, we first need to know the conditional PMF.

$$p_{X|Y}(x|10^2) = \left[\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{6} \quad 0 \quad 0 \right]$$
$$p_{X|Y}(x|10^4) = \left[0 \quad 0 \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{6} \right].$$

Therefore, the conditional expectations are

$$\mathbb{E}[X | Y = 10^2] = (\quad) \left(\frac{1}{2} \right) + (\quad) \left(\frac{1}{3} \right) + (\quad) \left(\frac{1}{6} \right) = \frac{123}{600}$$

$$\mathbb{E}[X | Y = 10^4] = (\quad) \left(\frac{1}{2} \right) + (\quad) \left(\frac{1}{3} \right) + (\quad) \left(\frac{1}{6} \right) = \frac{123}{6}.$$

From the conditional expectations we can also find $\mathbb{E}[X]$:

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X | Y = 10^2] (\quad) + \mathbb{E}[X | Y = 10^4] (\quad) \\ &= \left(\frac{123}{600} \right) (\quad) + \left(\frac{123}{6} \right) (\quad) = 3.5875. \end{aligned}$$

Example 3

Example 3. Consider two random variables X and Y .

- The random variable X is Gaussian distributed with $X \sim \text{Gaussian}(\mu, \sigma^2)$.
- The random variable Y has a conditional distribution $Y|X \sim \text{Gaussian}(X, X^2)$.

Find $\mathbb{E}[Y]$.

Solution.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \text{and}$$
$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi x^2}} e^{-\frac{(y-x)^2}{2x^2}}.$$

Example 3

The conditional expectation of Y given X is

$$\begin{aligned}\mathbb{E}[Y|X = x] &= \\ &= \mathbb{E}[\text{Gaussian}(x, x^2)] = x.\end{aligned}$$

So,

$$\begin{aligned}\mathbb{E}[Y] &= \\ &= \mathbb{E}[\text{Gaussian}(\mu, \sigma^2)] = \mu\end{aligned}$$

Summary

Questions?