

# ECE 302: Lecture 5.4 Conditional Distributions

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# Outline

- Joint PDF and CDF
- Joint Expectation
- **Conditional Distribution**
- Conditional Expectation
- Sum of Two Random Variables
- Random Vectors
- High-dimensional Gaussians and Transformation
- Principal Component Analysis

## Today's lecture

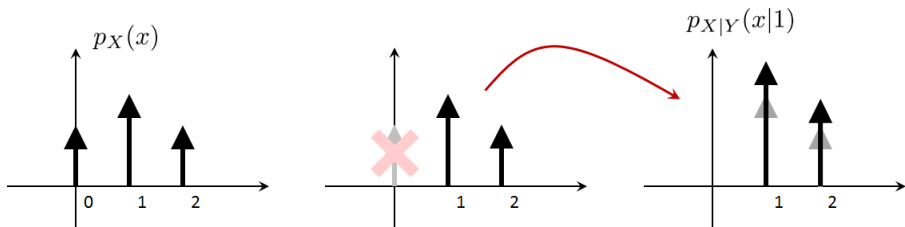
- Joint PMF
- Joint PDF
- Decomposition – Law of total probability

# Conditional PMF

## Definition

Let  $X$  and  $Y$  be two discrete random variables. The **conditional PMF** of  $X$  given  $Y$  is

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}. \quad (1)$$



**Figure:** Suppose  $X$  is the sum of two coins, and  $Y$  is the first coin. When  $X$  is unconditioned, the PMF is just  $p_X(x)$ . When  $X$  is conditioned on  $Y = 1$ , then  $X = 0$  cannot happen. Therefore, the resulting PMF  $p_{X|Y}(x|1)$  only has two states. After normalization we obtain the conditional PMF.

## Example 1

Consider a joint PMF given in the following table. Find the conditional PMF  $p_{X|Y}(x|1)$  and the marginal PMF  $p_X(x)$ .

	Y=			
	1	2	3	4
X = 1	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{0}{20}$
2	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{1}{20}$
3	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{1}{20}$
4	$\frac{0}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$

**Solution:**

$$p_X(x) = \left[ \frac{3}{20} \quad \frac{6}{20} \quad \frac{8}{20} \quad \frac{3}{20} \right]$$
$$p_{X|Y}(x|1) = \frac{p_{X,Y}(x,1)}{p_Y(1)} = \frac{\left[ \frac{1}{20} \quad \frac{1}{20} \quad \frac{1}{20} \quad \frac{0}{20} \right]}{\frac{3}{20}} = \left[ \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \quad 0 \right].$$

## Example 2

Consider two random variables  $X$  and  $Y$  defined as follows.

$$Y = \begin{cases} 10^2, & \text{with prob } 5/6, \\ 10^4, & \text{with prob } 1/6. \end{cases} \quad X = \begin{cases} 10^{-4}Y, & \text{with prob } 1/2, \\ 10^{-3}Y, & \text{with prob } 1/3, \\ 10^{-2}Y, & \text{with prob } 1/6. \end{cases}$$

Find  $p_{X|Y}(x|y)$ ,  $p_X(x)$  and  $p_{X,Y}(x,y)$ .

**Solution.**

$$p_{X|Y}(x|10^2) = \begin{cases} 1/2, & \text{if } x = 0.01, \\ 1/3, & \text{if } x = 0.1, \\ 1/6, & \text{if } x = 1. \end{cases}$$

$$p_{X|Y}(x|10^4) = \begin{cases} 1/2, & \text{if } x = 1, \\ 1/3, & \text{if } x = 10, \\ 1/6, & \text{if } x = 100. \end{cases}$$

## Example 2

Therefore, the joint PMF  $p_{X,Y}(x,y)$  is given by the following table.

$10^4$	0	0	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$
$10^2$	$\frac{5}{12}$	$\frac{5}{18}$	$\frac{5}{36}$	0	0
	0.01	0.1	1	10	100

The marginal PMF  $p_X(x)$  is

$$p_X(x) = \sum_y p_{X,Y}(x,y) = \left[ \frac{5}{12} \quad \frac{5}{18} \quad \frac{2}{9} \quad \frac{1}{18} \quad \frac{1}{36} \right].$$

## Example 3

Let us follow up on Example 1. What is the probability that  $\mathbb{P}[X > 2|Y = 1]$ ? What is the probability that  $\mathbb{P}[X > 2]$ ?

**Solution.** Since the problem asks about the conditional probability, we know that it can be computed by using the conditional PMF. This gives us

$$\begin{aligned}\mathbb{P}[X > 2|Y = 1] &= \sum_{x>2} p_{X|Y}(x|1) \\ &= \cancel{p_{X|Y}(1|1)} + \cancel{p_{X|Y}(2|1)} + \underbrace{p_{X|Y}(3|1)}_{\frac{1}{3}} + \underbrace{p_{X|Y}(4|1)}_0 = \frac{1}{3}.\end{aligned}$$

The other probability is

$$\mathbb{P}[X > 2] = \sum_{x>2} p_X(x) = \cancel{p_X(1)} + \cancel{p_X(2)} + \underbrace{p_X(3)}_{\frac{8}{20}} + \underbrace{p_X(4)}_{\frac{3}{20}} = \frac{11}{20}.$$

# Decomposition

## Theorem

Let  $X$  and  $Y$  be two discrete random variables, and let  $A$  be an event.  
Then

$$(i) \quad \mathbb{P}[X \in A | Y = y] = \sum_{x \in A} p_{X|Y}(x|y)$$

$$(ii) \quad \mathbb{P}[X \in A] = \sum_{x \in A} \sum_{y \in \Omega_Y} p_{X|Y}(x|y)p_Y(y) = \sum_{y \in \Omega_Y} \mathbb{P}[X \in A | Y = y]p_Y(y)$$

### What is the rule of thumb for conditional distribution?

- The PMF/PDF should *match* with the probability you are finding.
- If you are finding conditional probability  $\mathbb{P}[X \in A | Y = y]$ , then use the conditional PMF  $p_{X|Y}(x|y)$ .
- If you are finding the probability  $\mathbb{P}[X \in A]$ , then use the marginal PMF  $p_X(x)$ .



# Conditional PDF

## Definition

Let  $X$  and  $Y$  be two continuous random variables. The **conditional PDF** of  $X$  given  $Y$  is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}. \quad (2)$$

## Theorem

Let  $X$  and  $Y$  be continuous random variables. Let  $A$  be an event. Then

- (i)  $\mathbb{P}[X \in A | Y = y] = \int_A f_{X|Y}(x|y) dx$
- (ii)  $\mathbb{P}[X \in A] = \int_{\Omega_Y} \mathbb{P}[X \in A | Y = y] f_Y(y) dy.$

## Example 1

**Example 1.** Let  $X$  and  $Y$ :

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x}e^{-y}, & 0 \leq y \leq x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional PDF  $f_{X|Y}(x|y)$  and  $f_{Y|X}(y|x)$ .

**Solution.**

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy = \int_0^x 2e^{-x}e^{-y} dy = 2e^{-x}(1 - e^{-x})$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx = \int_y^{\infty} 2e^{-x}e^{-y} dx = 2e^{-2y}.$$

Therefore, the conditional PDFs are

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2e^{-x}e^{-y}}{2e^{-2y}} = e^{-(x+y)}, \quad x \geq y$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2e^{-x}e^{-y}}{2e^{-x}(1 - e^{-x})} = \frac{e^{-y}}{1 - e^{-x}}, \quad 0 \leq y < x.$$

## Example 2

**Example 2.** This example considers a classical detection problem. Let  $X$  be a random bit such that

$$X = \begin{cases} +1, & \text{with prob } 1/2 \\ -1, & \text{with prob } 1/2. \end{cases}$$

Suppose that  $X$  is transmitted over a noisy channel so that the observed signal is

$$Y = X + N,$$

where  $N \sim \text{Gaussian}(0, 1)$  is the noise which is independent to the signal  $X$ . Find the probability  $\mathbb{P}[X = +1 \mid Y > 0]$  and  $\mathbb{P}[X = -1 \mid Y > 0]$ .

## Example 2

**Solution.** First of all, we know that

$$f_{Y|X}(y|+1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}}, \quad \text{and} \quad f_{Y|X}(y|-1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2}}.$$

Therefore, integrating  $y$  from 0 to  $\infty$  will give us

$$\begin{aligned} \mathbb{P}[Y > 0 | X = +1] &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}} dy \\ &= 1 - \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}} dy \\ &= 1 - \Phi\left(\frac{0-1}{1}\right) = 1 - \Phi(-1). \end{aligned}$$

Similarly, we have  $\mathbb{P}[Y > 0 | X = -1] = 1 - \Phi(+1)$ .

By Bayes theorem,

$$\mathbb{P}[X = +1 | Y > 0] = \frac{\mathbb{P}[Y > 0 | X = +1]\mathbb{P}[X = +1]}{\mathbb{P}[Y > 0]}.$$

The denominator can be found by using the law of total probability:

$$\begin{aligned}\mathbb{P}[Y > 0] &= \mathbb{P}[Y > 0 | X = +1]\mathbb{P}[X = +1] + \mathbb{P}[Y > 0 | X = -1]\mathbb{P}[X = -1] \\ &= 1 - \frac{1}{2}(\Phi(+1) + \Phi(-1)) = \frac{1}{2},\end{aligned}$$

because  $\Phi(+1) + \Phi(-1) = \Phi(+1) + 1 - \Phi(+1) = 1$ . Therefore,

$$\mathbb{P}[X = +1 | Y > 0] = 1 - \Phi(-1) = 0.8413.$$

The implication is that if  $Y > 0$ , the probability

$$\begin{aligned}\mathbb{P}[X = +1 | Y > 0] &= 0.8413. \text{ The complement of this result gives that} \\ \mathbb{P}[X = -1 | Y > 0] &= 1 - 0.8413 = 0.1587.\end{aligned}$$

# Summary

## Definition

Let  $X$  and  $Y$  be two continuous random variables. The **conditional PDF** of  $X$  given  $Y$  is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}. \quad (3)$$

## Theorem

Let  $X$  and  $Y$  be continuous random variables. Let  $A$  be an event. Then

- (i)  $\mathbb{P}[X \in A | Y = y] = \int_A f_{X|Y}(x|y) dx$
- (ii)  $\mathbb{P}[X \in A] = \int_{\Omega_Y} \mathbb{P}[X \in A | Y = y] f_Y(y) dy.$

**Questions?**