ECE 302: Lecture 4.9 Generating Random Numbers

Prof Stanley Chan

School of Electrical and Computer Engineering
Purdue University

Outline

Overall schedule:

- Continuous random variables, PDF
- CDF
- Expectation
- Mean, mode, median
- Common random variables
  - Uniform
  - Exponential
  - Gaussian
- Transformation of random variables
- **How to generate random numbers**

Today's lecture:

- General principle
- Examples

Principle

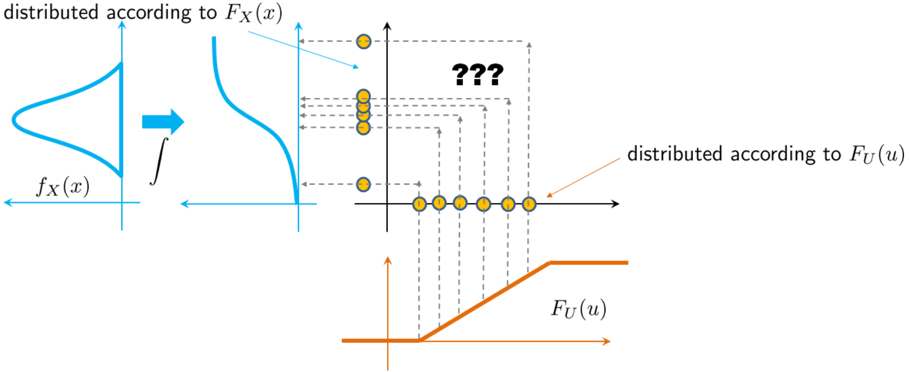


Figure: Generating random number according to a known CDF. The idea is to first generate a uniform(0,1) random variable, and then do an inverse mapping F_X^{-1}.

Main theorem

Theorem

The transformation is

$$g(u) = F_X^{-1}(u). \tag{1}$$

That is, if $g = F_X^{-1}$, then $g(U)$ will be distributed according to f_X (or F_X).

First, if $U \sim \mathrm{Uniform}(0,1)$, then $f_U(u) = 1$ for $0 \leq u \leq 1$ and so

$$F_U(u) = \int_{-\infty}^{u} f_U(u)du = u, \qquad 0 \leq u \leq 1.$$

Let $g = F_X^{-1}$ and define $Y = g(U)$. Then the CDF of Y is

$$\begin{aligned} F_Y(y) &= \mathbb{P}[Y \leq y] \\ &= \mathbb{P}[g(U) \leq y] \\ &= \mathbb{P}[F_X^{-1}(U) \leq y] \\ &= \mathbb{P}[U \leq F_X(y)] = F_X(y). \end{aligned}$$

Idea

How to generate random numbers from an arbitrary distribution F_X?

- Step 1: Generate a random number

$$U \sim \text{Uniform}(0, 1).$$

- Step 2: Let

$$Y = F_X^{-1}(U). \tag{2}$$

  Then the distribution of Y is F_X.

MATLAB: Suppose we are told that $F_X^{-1}(U) = -\frac{1}{\lambda}\log(1-U)$.

- Step 1: `U = rand(10000,1)`
- Step 2: `Y = -(1/lambda)*log(1-U)`

(Similar commands can be found in Python.)

Examples

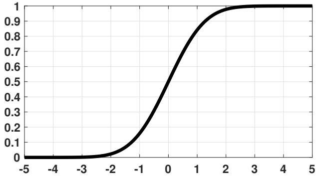
Example 1. How to generate, from uniform random numbers to, **Gaussian** random numbers with mean μ, and variance σ^2?

First of all, we generate $U \sim \text{Uniform}(0,1)$. The CDF of the ideal distribution is

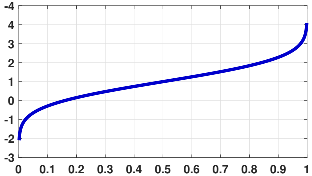
$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Therefore, the transformation g is

$$g(U) = F_X^{-1}(U) = \sigma\Phi^{-1}(U) + \mu.$$



(a) $F_X(x)$



(b) $g(U)$

Examples

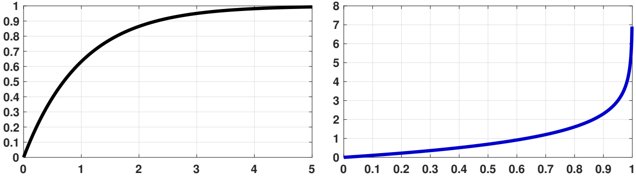
Example 2. How to generate, from uniform random numbers to, **exponential** random numbers with parameter λ?

First of all, we generate $U \sim \text{Uniform}(0,1)$. The CDF of the ideal distribution is

$$F_X(x) = 1 - e^{-\lambda x}.$$

Therefore, the transformation g is

$$g(U) = F_X^{-1}(U) = -\frac{1}{\lambda}\log(1-U).$$



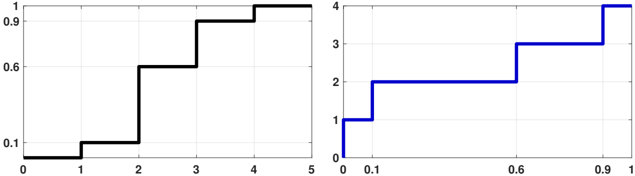
(a) $F_X(x)$ (b) $g(U)$

Examples

Example 3. How to generate, from uniform random numbers to, 4 numbers $1, 2, 3, 4$ according to the histogram [0.1 0.5 0.3 0.1].

First of all, we generate $U \sim \text{Uniform}(0, 1)$. The CDF of the ideal distribution is

$$F_X(x) = \begin{cases} 0.1, & x = 1, \\ 0.1 + 0.5 = 0.6, & x = 2, \\ 0.1 + 0.5 + 0.3 = 0.9, & x = 3, \\ 0.1 + 0.5 + 0.3 + 0.1 = 1.0, & x = 4. \end{cases}$$



(a) $F_X(x)$ (b) $g(U)$

Examples

This CDF is not invertible. However, we can still define the "inverse" mapping as

$$
\begin{aligned}
g(U) &= F_X^{-1}(U) \\
&= \begin{cases} 1, & 0.0 \leq U \leq 0.1, \\ 2, & 0.1 \leq U \leq 0.6, \\ 3, & 0.6 \leq U \leq 0.9, \\ 4, & 0.9 \leq U \leq 1.0. \end{cases}
\end{aligned}
$$

Summary

How to generate random numbers from an arbitrary distribution F_X?

- Step 1: Generate a random number

$$U \sim \mathrm{Uniform}(0, 1).$$

- Step 2: Let

$$Y = F_X^{-1}(U). \tag{3}$$

  Then the distribution of Y is F_X.

Questions?