ECE 302: Lecture 4.8 Function of Random Variables

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Outline

Overall schedule:

- Continuous random variables, PDF
- CDF
- Expectation
- Mean, mode, median
- Common random variables
  - Uniform
  - Exponential
  - Gaussian
- **Transformation of random variables**
- How to generate random numbers

Today's lecture:

- General principle
- Examples

The problem

You know X:

- PDF $f_X(x)$
- CDF $F_X(x)$

I tell you $Y = g(X)$ for some function g

I ask you to find Y:

- PDF $f_Y(y)$
- CDF $F_Y(y)$

Examples

Example 1. Let X be a random variable with PDF $f_X(x)$ and CDF $F_X(x)$. Let $Y = 2X + 3$. Find $f_Y(y)$ and $F_Y(y)$. Express answers in terms of $f_X(x)$ and $F_X(x)$.

Example 2. Let X be a random variable with PDF $f_X(x)$ and CDF $F_X(x)$. Suppose that $Y = X^2$, find $f_Y(y)$ and $F_Y(y)$. Express answers in terms of $f_X(x)$ and $F_X(x)$.

Example 3. Let $X \sim \text{Uniform}(0, 2\pi)$. Suppose $Y = \cos X$. Find $f_Y(y)$ and $F_Y(y)$.

Example 4. Let X be a random variable with PDF

$$f_X(x) = ae^x e^{-ae^x}.$$

Let $Y = e^X$, and find $f_Y(y)$.

Principle

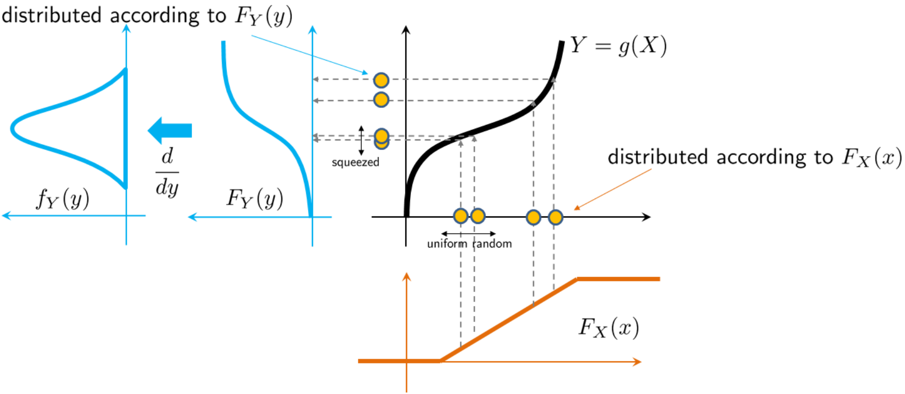

Figure: When transforming a random variable X to $Y = g(X)$, the distributions are defined according to the spacing between samples. In this figure, a uniformly distributed X will become squeezed by some parts of g, and widened in another part of g.

Example 1

Example 1. (Linear transform.) Let X be a random variable with PDF $f_X(x)$ and CDF $F_X(x)$. Let $Y = 2X + 3$. Find $f_Y(y)$ and $F_Y(y)$. Express answers in terms of $f_X(x)$ and $F_X(x)$.

$$F_Y(y) = \qquad\qquad = F_X\left(\frac{y-3}{2}\right).$$

$$f_Y(y) = \qquad\qquad = \frac{1}{2} f_X\left(\frac{y-3}{2}\right).$$

Example 2

Let X be a random variable with PDF $f_X(x)$ and CDF $F_X(x)$. Suppose that $Y = X^2$, find $f_Y(y)$ and $F_Y(y)$. Express answers in terms of $f_X(x)$ and $F_X(x)$.

$$F_Y(y) = \qquad = F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

$$f_Y(y) = \qquad = \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right).$$

Example 2

Follow Up. (Square of a uniform random variable) Suppose X is a uniform random variable in $[a, b]$ (assume $a > 0$), and let $Y = X^2$, then the CDF and PDF of Y are respectively

$$F_Y(y) = \frac{\sqrt{y} - a}{b - a} - \frac{-\sqrt{y} - a}{b - a}, \qquad a^2 \leq y \leq b^2,$$
$$f_Y(y) = \frac{1}{\sqrt{y}(b - a)}, \qquad a^2 \leq y \leq b^2.$$

Example 2

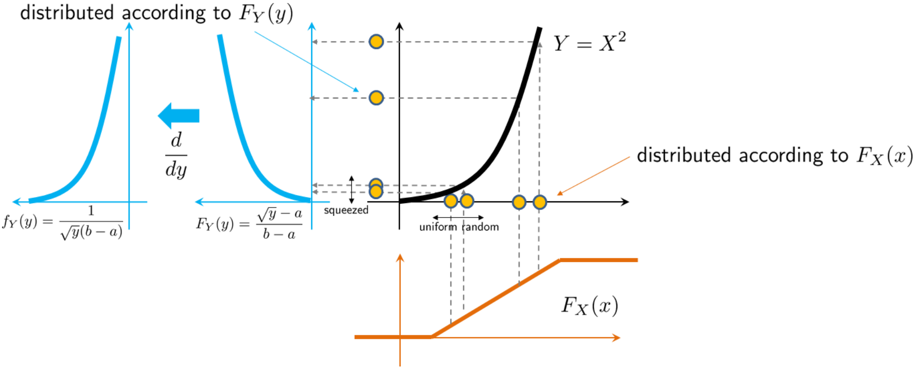

Figure: When transforming a random variable X to $Y = X^2$, the CDF becomes $F_Y(y) = \frac{\sqrt{y}-a}{b-a}$ and the PDF becomes $f_Y(y) = \frac{1}{\sqrt{y}(b-a)}$.

Example 3

Let $X \sim \mathrm{Uniform}(0, 2\pi)$. Suppose $Y = \cos X$. Find $f_Y(y)$ and $F_Y(y)$.

$$F_X(x) = \qquad\qquad = \frac{x}{2\pi}.$$

Thus, the CDF of Y is

$$\begin{aligned} F_Y(y) &= \\ &= \\ &= \\ &= F_X(2\pi - \cos^{-1} y) - F_X(\cos^{-1} y) \\ &= 1 - \frac{\cos^{-1} y}{\pi}. \end{aligned}$$

Example 3

Hint: $\frac{d}{dy}\cos^{-1} y = \frac{-1}{\sqrt{1-y^2}}$.

So, we have

$$\begin{aligned} f_Y(y) = \frac{d}{dy} F_Y(y) &= \frac{d}{dy}\left(1 - \frac{\cos^{-1} y}{\pi}\right) \\ &= \frac{1}{\pi\sqrt{1-y^2}} \end{aligned}$$

Example 4

Let X be a random variable with PDF

$$f_X(x) = ae^x e^{-ae^x}.$$

Let $Y = e^X$, and find $f_Y(y)$.

$$\begin{aligned} F_Y(y) &= \\ &= \\ &= \\ &= \int_{-\infty}^{\log y} ae^x e^{-ae^x} \, dx. \end{aligned}$$

Example 4

$$\begin{aligned} f_Y(y) &= \\ &= \\ &= \\ &= ae^{-ay}. \end{aligned}$$

Summary

Steps

- Step 1: Find the CDF $F_Y(y)$.
- Step 2: Find the PDF $f_Y(y)$.

Rules of thumbs

- Always find the CDF $F_Y(y) = \mathbb{P}[g(X) \leq y]$. Ask yourself: What are the X such that $g(X) \leq y$? Think of the cosine example.
- Sometimes you do not need to explicitly solve for $F_Y(y)$. The fundamental theorem of calculus can help you find $f_Y(y)$.
- Draw pictures. Ask yourself whether you need to squeeze or stretch the samples.

Questions?