ECE 302: Lecture 4.1 Probability Density Functions

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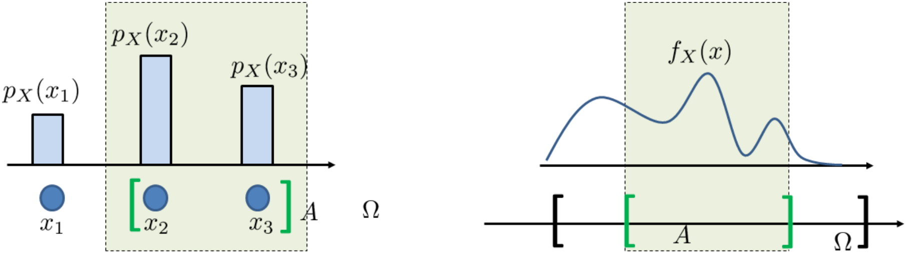
How to define probability for continuous events?



Outline

Today's lecture: Understand probability density functions (PDFs).

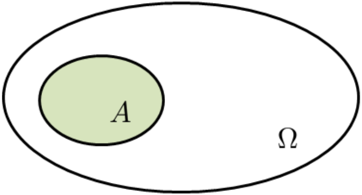
- Intuition
- Definition and properties
- Connecting with PMF

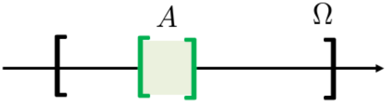
Intuition

How would you define $\mathbb{P}[\{x \in A\}]$?

Measure the size of a set:

$$\mathbb{P}[\{x \in A\}] = \frac{\text{“size” of } A}{\text{“size” of } \Omega}. \tag{1}$$





Example

Suppose that the sample space is the interval $\Omega = [0, 5]$ and the event is $A = [2, 3]$. To measure the "size" of A, we can integrate A to determine the length. That is,

$$\mathbb{P}[\{x \in [2,3]\}] = \frac{\text{“size” of } A}{\text{“size” of } \Omega} = \frac{\int_A dx}{\int_\Omega dx} = \frac{\int_2^3 dx}{\int_0^5 dx} = \frac{1}{5}. \tag{2}$$

More formally,

$$\begin{aligned} \mathbb{P}[\{x \in A\}] &= \frac{\int_A dx}{\int_\Omega dx} = \frac{\int_A dx}{|\Omega|} \\ &= \int_A \underbrace{\frac{1}{|\Omega|}}_{\text{equally important over } \Omega} dx. \end{aligned} \tag{3}$$

Relax equiprobable

What happens if we want to relax the "equiprobable" assumption?

Replace the constant function $1/|\Omega|$ with $f_X(x)$. This will give us

$$\mathbb{P}[\{x \in A\}] = \int_A \underbrace{f_X(x)}_{\text{replace } 1/|\Omega|} dx. \tag{4}$$

If you compare it with a PMF, we note that when X is discrete, then

$$\mathbb{P}[\{x \in A\}] = \sum_{x \in A} p_X(x).$$

Probability density function

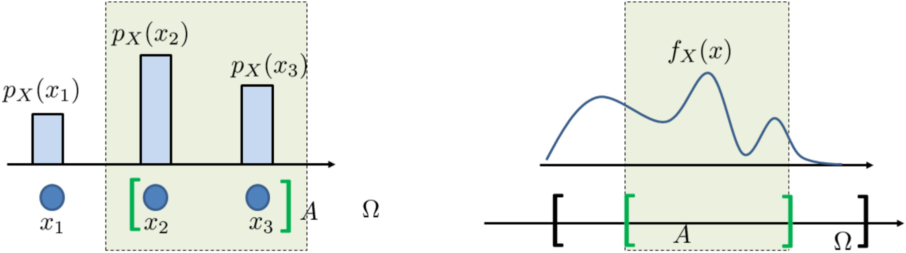


Figure: [Left] A probability mass function (PMF) tells us the relative frequency of a state when computing the probability. In this example, the "size" of A is $p_X(x_2) + p_X(x_3)$. [Right] A probability density function (PDF) is the infinitesimal version of the PMF. Thus, the "size" of A is the integration over the PDF.

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Definition

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A probability density function f_X of a random variable X is a mapping $f_X : \Omega \to \mathbb{R}$, with the property that

- Non-negativity: $f_X(x) \geq 0$ for all $x \in \Omega$
- Unity: $\int_\Omega f_X(x) dx = 1$
- Measure of a set: $\mathbb{P}[\{x \in A\}] = \int_A f_X(x) dx$

Definition

Definition

Let X be a continuous random variable. The probability density function (PDF) of X is a function $f_X : \Omega \to \mathbb{R}$, when integrated over an interval $[a, b]$, yields the probability of obtaining $a \leq X \leq b$:

$$\mathbb{P}[a \leq X \leq b] = \int_a^b f_X(x)dx. \tag{5}$$

Examples

Example 1. Let $f_X(x) = 3x^2$ with $\Omega = [0,1]$. Let $A = [0, 0.5]$. Then, the probability $\mathbb{P}[\{X \in A\}]$ is

$$\mathbb{P}[0 \leq X \leq 0.5] = \int_0^{0.5} 3x^2 dx = \frac{1}{8}. \quad \square$$

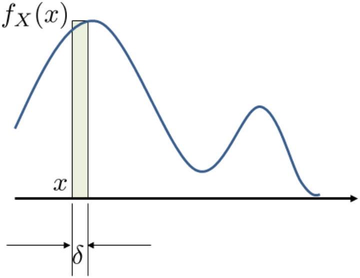
Example 2. Let $f_X(x) = 1/|\Omega|$ with $\Omega = [0,5]$. Let $A = [3,5]$. Then, the probability $\mathbb{P}[\{X \in A\}]$ is

$$\mathbb{P}[3 \leq X \leq 5] = \int_3^{5} \frac{1}{|\Omega|} dx = \int_3^5 \frac{1}{5} dx = \frac{2}{5}. \quad \square$$

Can $f_X(x) \geq 1$?

- Yes.
- $f_X(x)$ is *not* the probability of having $X = x$
- $f_X(x)$ is the probability *per unit length*

$$\mathbb{P}[x \leq X \leq x + \delta] = \int_x^{x+\delta} f_X(x)dx \approx f_X(x) \cdot \delta. \quad (6)$$



Examples

Example. Consider a random variable X with PDF $f_X(x) = \frac{1}{2\sqrt{x}}$ for any $0 < x \leq 1$, and is 0 otherwise. We can show that $f_X(x) \to \infty$ as $x \to 0$. However, $f_X(x)$ remains a valid PDF because

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^1 \frac{1}{2\sqrt{x}}dx = \sqrt{x}\Big|_0^1 = 1. \quad \square$$

Remark. Since isolated points have zero measure in the continuous space, the probability of an open interval (a, b) is exactly the same as the probability of a closed interval:

$$\mathbb{P}[[a, b]] = \mathbb{P}[(a, b)] = \mathbb{P}[(a, b]] = \mathbb{P}[[a, b)].$$

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How to write a PMF in terms of a PDF?

Use delta function:

$$f_X(x) = \sum_{x_k \in \Omega} p_X(x_k)\delta(x - x_k). \tag{7}$$

Examples

Example 1. If X is a Bernoulli random variable with PMF $p_X(1) = p$ and $p_X(0) = 1 - p$, then the corresponding PDF can be written as

$$f_X(x) = p\delta(x-1) + (1-p)\delta(x-0). \qquad \square$$

Example 2. If X is a binomial random variable with PMF $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, then the corresponding PDF can be written as

$$f_X(x) = \sum_{k=0}^{n} p_X(k)\delta(x-k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(x-k). \qquad \square$$

Example

Example 3. Let X be a discrete random variable with PMF

$$p_X(k) = \frac{1}{2^k}, \qquad k = 1, 2, \ldots,$$

The continuous representation of the PMF can be written as

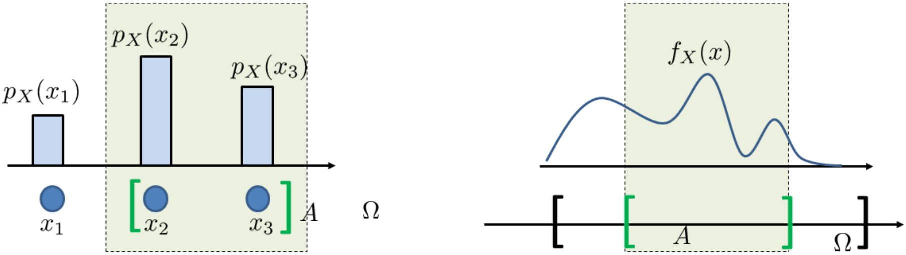
$$f_X(x) = \sum_{k=1}^{\infty} p_X(k)\delta(x-k) = \sum_{k=1}^{\infty} \left(\frac{1}{2^k}\right) \delta(x-k).$$

Find the probability $\mathbb{P}[1 \leq X \leq 2]$.

Example

$$
\begin{aligned}
&\mathbb{P}[1 \leq X \leq 2] \\
&= \int_1^2 f_X(x)dx = \int_1^2 \sum_{k=1}^{\infty} \left(\frac{1}{2^k}\right) \delta(x-k)dx \\
&= \int_1^2 \left\{ \frac{1}{2}\delta(x-1) + \frac{1}{4}\delta(x-2) + \ldots \right\} dx \\
&= \frac{1}{2}\underbrace{\int_1^2 \delta(x-1)dx}_{=1} + \frac{1}{4}\underbrace{\int_1^2 \delta(x-2)dx}_{=1} + \frac{1}{8}\underbrace{\int_1^2 \delta(x-3)dx}_{=0} + \underbrace{\ldots}_{=0} \\
&= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.
\end{aligned}
$$

Summary



- **Intuition**
  - Probability is a measure of the size of a set
  - Use length/area/volume to measure the size of a continuous set
  - $f_X(x)$ is the weight when calculating the size
- **Definition and properties**
  - Probability per unit length
  - $f_X(x) \geq 1$ is okay
- **Connecting with PMF**
  - PMF is a train of delta function

Questions?