Motivating Story: Cameras

\[ X = \text{number of photons} \]

\[ \lambda = \alpha t \]

\[ \mathbb{P}[X = k] = \frac{\lambda^k}{k!} e^{-\lambda} \]
Mysteries about Poisson Random Variables

If you have learned Poisson random variables before,

- Your teacher probably asked you to memorize the PMF:

\[ p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 1, 2, \ldots.\]

- And then you will learn how to compute the mean and variance.
- Have you ever wondered why the Poisson PMF is defined in that way?
- Is there a principled way of deriving this formula?
- Besides the formula, what else can Poisson buy you?
- Why are Poisson random variables so important for computer vision, especially low-light imaging?
- We are going to tell you all these in today’s lecture.
3.1 Random variables
3.2 Probability mass functions (PMF)
3.3 Cumulative distribution functions (discrete case)
3.4 Expectation
3.5 Moments and variance
3.6 Bernoulli random variables
3.7 Binomial random variables
3.8 Geometric random variables
3.9 Poisson random variables
   - Definition of Poisson
   - Demystifying the mean and variance
   - Origin of Poisson
   - Poisson approximation to Binomial
Definition

Let $X$ be a **Poisson** random variable. Then, the PMF of $X$ is

$$p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 1, 2, \ldots,$$

where $\lambda > 0$ is the Poisson rate. We write

$$X \sim \text{Poisson}(\lambda)$$

to say that $X$ is drawn from a Poisson distribution with a parameter $\lambda$.

**Understanding the parameter:**
- $X =$ number of arrivals
- $\alpha =$ arrival rate = number per unit time
- $t =$ time
- So, $\lambda = \alpha t =$ average number within $t$ unit time.
The CDF of Poisson is

\[ F_X(k) = \mathbb{P}[X \leq k] = \sum_{\ell=0}^{k} \frac{\lambda^\ell}{\ell!} e^{-\lambda} . \]  

(1)
Example 1. In a telephone call center, the average number of phone calls is \( \lambda \) calls. Let \( X \) be the actual number of calls. Find \( P[X > 4] \) and \( P[X \leq 5] \).

Solution. By using a Poisson random variable with parameter \( \lambda \), we can show that the probabilities are

\[
P[X > 4] = 1 - P[X \leq 4] = 1 - \sum_{k=0}^{4} \frac{\lambda^k}{k!} e^{-\lambda},
\]

\[
P[X \leq 5] = \sum_{k=0}^{5} \frac{\lambda^k}{k!} e^{-\lambda}.
\]

Remark:
- \( \lambda \) = average number of calls
- E.g., \( \lambda = 1 \) can be achieved by \( \alpha = 0.5 \) and \( t = 2 \), or \( \alpha = 3 \) and \( t = 0.33 \).
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Mean and Variance

Proposition

If $X \sim \text{Poisson}(\lambda)$, then

$$E[X] = \lambda, \quad E[X^2] = \lambda + \lambda^2, \quad \text{Var}[X] = \lambda.$$
Interpreting the Mean and Variance

Proposition

If $X \sim \text{Poisson}(\lambda)$, then

$$
\mathbb{E}[X] = \lambda, \quad \mathbb{E}[X^2] = \lambda + \lambda^2, \quad \text{Var}[X] = \lambda.
$$

How do we understand the mean and the variance?

- Mean and variance are both $\lambda$
- Larger mean means higher variance!
- Brighter pixel means more noise!?
Wait a minute ...

This is what we get!

(a) $\lambda \in [0, 1]$  
(b) $\lambda \in [0, 10]$  
(c) $\lambda \in [0, 100]$  

Inconsistent with “brighter pixel means more noise”.

Interpreting the Mean and Variance

Let $\lambda = \text{image intensity of one pixel}$, and let $X \sim \text{Poisson}(\lambda)$.

- It is true that “brighter pixel means more noise”.

But this is not what you care! What you care is the **signal-to-noise ratio** (SNR).

$$\text{SNR} = \frac{\mathbb{E}[X]}{\sqrt{\text{Var}[X]}} = \frac{\lambda}{\sqrt{\lambda}} = \sqrt{\lambda}.$$  

Brighter pixel means more noise, but your signal is also stronger. Gain in signal overwhelms the gain in noise. So still a good deal!
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The origin of Poisson random variable

**Question:**
- We knew how to derive Bernoulli, Binomial and Geometric. How about Poisson?
- Where does the Poisson formula come from?

**Why is this question so important?**
- It tells your the intuition of Poisson.
- It helps you to formulate your next engineering problem.
- A lot of textbooks ignore this. A big mistake.

**Our approach:**
- Use our camera story
- Argument based on physics
- Very intuitive
- No jargon, no bubble
Elementary Assumptions

- Let $\Delta t$ be very small. Assume

$$\mathbb{P}[X(t + \Delta t) - X(t) = 1] = \alpha \Delta t.$$ 

- Let $\Delta t$ be very small. Assume no more than 1 electron arrives

$$\mathbb{P}[X(t + \Delta t) - X(t) = 0] = 1 - \alpha \Delta t.$$ 

- The number of impulses in non-overlapping time intervals are **independent**.
Let us zoom in to $\Delta t$

$$
\mathbb{P}[X(t + \Delta t) = k] = \mathbb{P}[X(t) = k - 1] \cdot \mathbb{P}[\text{there is one photon in } \Delta t] = \alpha \Delta t + \mathbb{P}[X(t) = k] \cdot \mathbb{P}[\text{there is no photon in } \Delta t] = 1 - \alpha \Delta t
$$

$$
= \mathbb{P}[X(t) = k - 1] \cdot (\alpha \Delta t) + \mathbb{P}[X(t) = k] \cdot (1 - \alpha \Delta t)
$$

$$
= \mathbb{P}[X(t) = k] - \mathbb{P}[X(t) = k] \alpha \Delta t + \mathbb{P}[X(t) = k - 1] \alpha \Delta t.
$$

This will give us

$$
\frac{\mathbb{P}[X(t + \Delta t) = k] - \mathbb{P}[X(t) = k]}{\Delta t} = \alpha \left( \mathbb{P}[X(t) = k - 1] - \mathbb{P}[X(t) = k] \right).
$$
Setting the limiting of $\Delta t \to 0$, we arrive at an ordinary differential equation

$$\frac{d}{dt} \mathbb{P}[X(t) = k] = \alpha \left( \mathbb{P}[X(t) = k - 1] - \mathbb{P}[X(t) = k] \right). \quad (2)$$

Solution to this differential equation is

$$\mathbb{P}[X(t) = k] = \frac{(\alpha t)^k}{k!} e^{-\alpha t}$$

Why?

$$\frac{d}{dt} \mathbb{P}[X(t) = k] = \frac{d}{dt} \left( \frac{(\alpha t)^k}{k!} e^{-\alpha t} \right) = \alpha k \frac{(\alpha t)^{k-1}}{k!} e^{-\alpha t} + (-\alpha) \frac{(\alpha t)^k}{k!} e^{-\alpha t}$$

$$= \alpha \frac{(\alpha t)^{k-1}}{k!} e^{-\alpha t} - \alpha \frac{(\alpha t)^k}{k!} e^{-\alpha t}$$

$$= \alpha \left( \mathbb{P}[X(t) = k - 1] - \mathbb{P}[X(t) = k] \right),$$
We found it

So we have found the PMF:

$$\mathbb{P}[X(t) = k] = \frac{\alpha t^k}{k!} e^{-\alpha t}$$ (3)

- $\alpha = \text{rate} = \text{number per unit time}$
- $t = \text{time}$
- $\lambda = \alpha t = \text{average number}$

Replacing $\lambda = \alpha t$, we obtain

$$\mathbb{P}[X = k] = \frac{\lambda^k}{k!} e^{-\lambda}$$
Outline

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Another very important result:
- When \( N \) is large, binomial is approximately Poisson

Another way to derive the Poisson formula:
- Pure algebraic derivation
- Also correct
- Also easy to show
- Although the physics-based approach offers more intuition
Proposition

**Poisson Approximation to Binomial.** For small \( p \) and large \( n \),

\[
\binom{n}{k} p^k (1 - p)^{n-k} \approx \frac{\lambda^k}{k!} e^{-\lambda},
\]

where \( \lambda \overset{\text{def}}{=} np \).
Example

Problem.

- Data arrival rate: $n = 10^9$ bits per second.
- Probability of having one error bit: $p = 10^{-9}$.
- In one second, how likely will we get $k = 5$ error bits?

Solution.

If you stick to binomial:

- Binomial: Flip coin $10^9$ times. Get 5 heads.
- $\binom{10^9}{5}(10^{-9})^5(1 - 10^{-9})^{10^9-5}$.
- Numerical problem!!

If we use Poisson approximation:

- $\lambda = np = (10^9)(10^{-9}) = 1$.
- $\frac{1^5}{5!}e^{-1}$.
- Much easier!
Proof.

Let $\lambda = np$. Then,

\[
\binom{n}{k} p^k (1 - p)^{n-k}
\]

\[
= \frac{n!}{k!(n-k)!} \left( \frac{\lambda}{n} \right)^k \left( 1 - \frac{\lambda}{n} \right)^{n-k}
\]

\[
= \frac{\lambda^k}{k!} \frac{n(n-1) \ldots (n-k+1)}{n \cdot n \cdot \ldots \cdot n} \left( 1 - \frac{\lambda}{n} \right)^{n-k}
\]

\[
= \frac{\lambda^k}{k!} \left( 1 - \frac{1}{n} \right) \ldots \left( 1 - \frac{k-1}{n} \right) \left( 1 - \frac{\lambda}{n} \right)^{n-k} \left( 1 - \frac{\lambda}{n} \right)^n
\]

\[
\rightarrow 1 \text{ as } n \to \infty
\]

\[
= \frac{\lambda^k}{k!} \left( 1 - \frac{\lambda}{n} \right)^n
\].
Proof.

We claim that \((1 - \frac{\lambda}{n})^n \to e^{-\lambda}\). This can be proved by noting that

\[
\log(1 + x) \approx x, \quad x \ll 1.
\]

It then follows that \(
\log \left(1 - \frac{\lambda}{n}\right) \approx -\frac{\lambda}{n}.
\)

Hence, \((1 - \frac{\lambda}{n})^n \approx e^{-\lambda}\)
Questions?