ECE 302: Lecture 3.8 Geometric Random Variables

Prof Stanley Chan

School of Electrical and Computer Engineering
Purdue University
3.1 Random variables
3.2 Probability mass functions (PMF)
3.3 Cumulative distribution functions (discrete case)
3.4 Expectation
3.5 Moments and variance
3.6 Bernoulli random variables
3.7 Binomial random variables
3.8 Geometric random variables
3.9 Poisson random variables
Flipping a coin

Figure: Suppose you flip a coin until you see a head. This requires you to have $N - 1$ tails, and then followed by a head in the $N$-th coin flip. The probability of this sequence of events are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\ldots$, which forms an infinite series.
Geometric Random Variable

Definition
Let $X$ be a **Geometric** random variable. Then, the PMF of $X$ is

$$p_X(k) = (1 - p)^{k-1} p, \quad k = 1, 2, \ldots,$$

where $0 < p < 1$ is the Geometric parameter. We write

$$X \sim \text{Geometric}(p)$$

to say that $X$ is drawn from a Geometric distribution with a parameter $p$.

**Interpretation:** Flip a coin until you get a head.

$$p_X(k) = \underbrace{(1 - p)^{k-1}}_{k - 1 \text{ failures}} \underbrace{p}_{\text{final success}}.$$
Probability mass function

Figure: The histogram of flipping a coin until we see a Head. The x-axis denotes the number of coins we need to flip, and the y-axis denotes the probability.
Probability of success

If \( p = \frac{1}{2} \), then

\[
\begin{align*}
\mathbb{P}[\text{success after 1 attempt}] &= \frac{1}{2} = 0.5 \\
\mathbb{P}[\text{success after 2 attempts}] &= \frac{1}{2} + \frac{1}{4} = 0.75 \\
\mathbb{P}[\text{success after 3 attempts}] &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875 \\
\mathbb{P}[\text{success after 4 attempts}] &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 0.9375.
\end{align*}
\]
Shape of a Geometric Random Variable

Table: PMFs of a geometric random variable $X \sim \text{Geometric}(p)$. 
Theorem

*The CDF of a geometric random variable is*

\[ F_X(k) = 1 - (1 - p)^k. \]  \hspace{1cm} (1)
Moments of Geometric Random Variables

Property

If $X \sim \text{Geometric}(p)$, then

\[
\mathbb{E}[X] = \frac{1}{p},
\]
\[
\mathbb{E}[X^2] = \frac{2}{p^2} - \frac{1}{p},
\]
\[
\text{Var}[X] = \frac{1 - p}{p^2}.
\]
Largest variance happens when ...

Since $\text{Var}[X] = \frac{1-p}{p^2}$, when will you have the largest variance?
Questions?