ECE 302: Lecture 3.7 Binomial Random Variables

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Data coming from a binary image sensor

In 2005, a new type of image sensor was proposed. The sensor is called the Quanta Image Sensor.

Every pixel is binary: Either 1 or 0. Probability of getting a 1 is $p$. The sensor can buy you...
The power of quanta image sensors


Fig. 10. Real Image Results. This figure shows raw Bayer data obtained from a prototype QIS and a commercially available CIS, and how they are classified using our proposed classifier. The inset images show the denoised images (by [43]) for visualization. Notice the heavy noise at 0.25 and 0.5 ppp, only QIS plus our proposed classification method can produce the correct prediction.
One basic question is:

- I have observed 100 frames.
- Since the pixels are binary, I can count the number of 1’s and 0’s for each pixel.
- What is the statistics of these 1’s and 0’s?
Outline

- 3.1 Random variables
- 3.2 Probability mass functions (PMF)
- 3.3 Cumulative distribution functions (discrete case)
- 3.4 Expectation
- 3.5 Moments and variance
- 3.6 Bernoulli random variables
- 3.7 Binomial random variables
  - Definition of binomial random variables
  - Relationship with Bernoulli
  - Expectation and variance
  - Application: Binary image sensors
- 3.8 Geometric random variables
- 3.9 Poisson random variables
**Binomial Random Variable**

**Definition**

Let $X$ be a **Binomial** random variable. Then, the PMF of $X$ is

$$p_X(k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots, n,$$

where $0 < p < 1$ is the Binomial parameter, and $n$ is the total number of states. We write

$$X \sim \text{Binomial}(n, p)$$

to say that $X$ is drawn from a Binomial distribution with a parameter $p$ of size $n$.

**Example.** Number of heads in $n$ coin flips.
Origin of binomial random variables

Flip a coin 3 times. Find the probability of getting 3 heads.

\[ p_X(3) = \mathbb{P}\left[ \{ “HHH” \} \right] = \mathbb{P}\left[ \{ “H” \} \cap \{ “H” \} \cap \{ “H” \} \right] \]

\[ = \mathbb{P}\left[ \{ “H” \} \right] \mathbb{P}\left[ \{ “H” \} \right] \mathbb{P}\left[ \{ “H” \} \right] \]

\[ \Rightarrow (b) \mathbb{P}\left[ \{ “H” \} \right] \mathbb{P}\left[ \{ “H” \} \right] \mathbb{P}\left[ \{ “H” \} \right] = p^3, \]

Find the probability of getting 2 heads.

\[ p_X(2) = \mathbb{P}\left[ \{ “HHT” \} \cup \{ “HTH” \} \cup \{ “THH” \} \right] \]

\[ \Rightarrow (c) \mathbb{P}\left[ \{ “HHT” \} \right] + \mathbb{P}\left[ \{ “HTH” \} \right] + \mathbb{P}\left[ \{ “THH” \} \right] \]

\[ = p^2(1 - p) + p^2(1 - p) + p^2(1 - p) = 3p^2(1 - p), \]
Origin of binomial random variables

In general,

\[ p_X(k) = \binom{n}{k} p^k (1 - p)^{n-k} \]  

(number of combinations) prob getting \( k \) H's (prob getting \( n - k \) T's)
Table: PMFs of a binomial random variable $X \sim \text{Binomial}(n, p)$. 

(a) $n = 60$ 

(b) $p = 0.5$
Moments of Binomial

Property

If $X \sim \text{Binomial}(n, p)$, then

\[ \mathbb{E}[X] = np, \]
\[ \mathbb{E}[X^2] = np(np + (1 - p)), \]
\[ \text{Var}[X] = np(1 - p). \]

Proof.

\[
\mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1 - p)^{n-k} = \sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k (1 - p)^{n-k} \\
= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1 - p)^{n-k}. 
\]

... a few more steps.
A short cut to the proof
PMF and CDF

\[ F_X(k) = \sum_{\ell=0}^{k} \binom{k}{\ell} p^\ell (1-p)^{k-\ell}. \]  \hspace{1cm} (2)

**Table: PMF and CDF of a binomial random variable** \( X \sim \text{Binomial}(n, p) \).
Going back to the binary sensor...

How to model the random variable \( X = \text{number of 1's observed in 100 measurements?} \)
Questions?