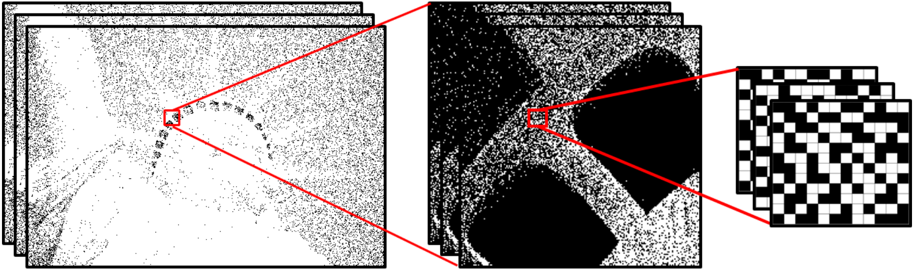
ECE 302: Lecture 3.7 Binomial Random Variables

Prof Stanley Chan

School of Electrical and Computer Engineering
Purdue University

Data coming from a binary image sensor

In 2005, a new type of image sensor was proposed. The sensor is called the Quanta Image Sensor.



Every pixel is binary: Either 1 or 0. Probability of getting a 1 is p.
The sensor can buy you...

The power of quanta image sensors

Abhiram Gnanasambandam, Stanley H. Chan, “Image Classification in the Dark using Quanta Image Sensors” European Conference on Computer Vision (ECCV 2020)

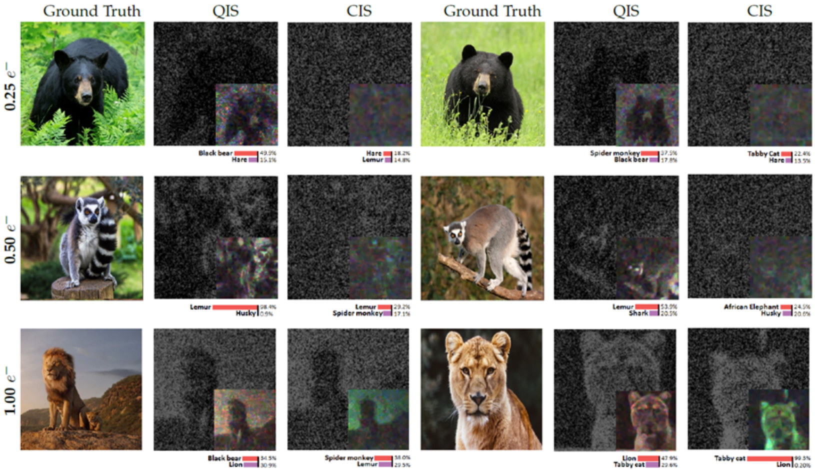
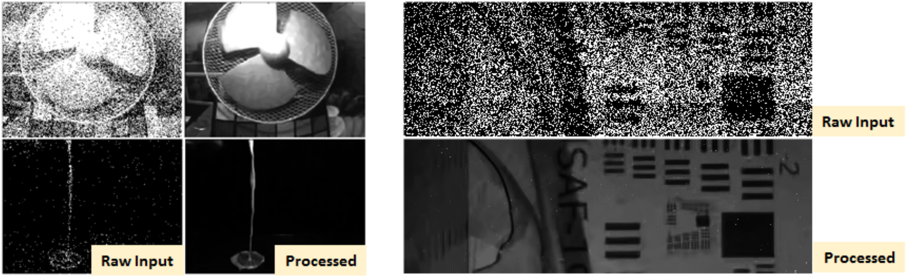


Fig. 10. **Real Image Results.** This figure shows raw Bayer data obtained from a prototype QIS and a commercially available CIS, and how they are classified using our proposed classifier. The inset images show the denoised images (by [43]) for visualization. Notice the heavy noise at 0.25 and 0.5 ppp, only QIS plus our proposed classification method can produce the correct prediction.

The power of quanta image sensors



Stanley H. Chan, Omar Elgendy and Xiran Wang, "**Images from bits: Non-iterative image reconstruction for quanta image sensors**", MDPI Sensors Special Issue on Photon-Counting Image Sensors, vol. 16, no. 11, paper 1961, pp.1-21, Nov. 2016. 7

One basic question is:

- I have observed 100 frames.
- Since the pixels are binary, I can count the number of 1's and 0's for each pixel.
- What is the statistics of these 1's and 0's?

Outline

Binomial Random Variable

Definition

Let X be a **Binomial** random variable. Then, the PMF of X is

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n,$$

where $0 < p < 1$ is the Binomial parameter, and n is the total number of states. We write

$$X \sim \text{Binomial}(n, p)$$

to say that X is drawn from a Binomial distribution with a parameter p of size n.

Example. Number of heads in n coin flips.

Origin of binomial random variables

Flip a coin 3 times. Find the probability of getting 3 heads.

$$
\begin{aligned}
p_X(3) = \mathbb{P}[\{\text{“HHH”}\}] &= \mathbb{P}[\{\text{“H”}\} \cap \{\text{“H”}\} \cap \{\text{“H”}\}] \\
&\overset{(a)}{=} \mathbb{P}[\{\text{“H”}\}]\mathbb{P}[\{\text{“H”}\}]\mathbb{P}[\{\text{“H”}\}] \overset{(b)}{=} p^3,
\end{aligned}
$$

Find the probability of getting 2 heads.

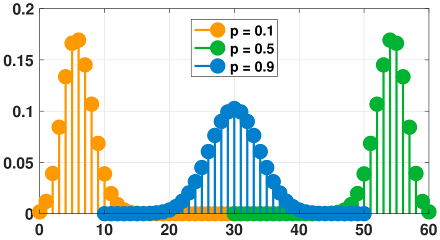
$$
\begin{aligned}
p_X(2) &= \mathbb{P}[\{\text{“HHT”}\} \cup \{\text{“HTH”}\} \cup \{\text{“THH”}\}] \\
&\overset{(c)}{=} \mathbb{P}[\{\text{“HHT”}\}] + \mathbb{P}[\{\text{“HTH”}\}] + \mathbb{P}[\{\text{“THH”}\}] \\
&= p^2(1-p) + p^2(1-p) + p^2(1-p) = 3p^2(1-p),
\end{aligned}
$$

Origin of binomial random variables

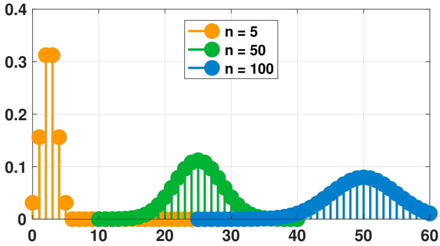
In general,

$$p_X(k) = \underbrace{\binom{n}{k}}_{\text{number of combinations}} \underbrace{p^k}_{\text{prob getting } k \text{ H's}} \underbrace{(1-p)^{n-k}}_{\text{prob getting } n-k \text{ T's}}. \quad (1)$$

Shape of a binomial PMF



(a) $n = 60$



(b) $p = 0.5$

Table: PMFs of a binomial random variable $X \sim \text{Binomial}(n, p)$.

Moments of Binomial

Property

If $X \sim \mathrm{Binomial}(n, p)$, *then*

$$\begin{aligned} \mathbb{E}[X] &= np, \\ \mathbb{E}[X^2] &= np(np + (1-p)), \\ \mathrm{Var}[X] &= np(1-p). \end{aligned}$$

Proof.

$$\begin{aligned} \mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} &= \sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}. \end{aligned}$$

... a few more steps.

A short cut to the proof

PMF and CDF

$$F_X(k) = \sum_{\ell=0}^{k} \binom{k}{\ell} p^{\ell}(1-p)^{k-\ell}. \tag{2}$$

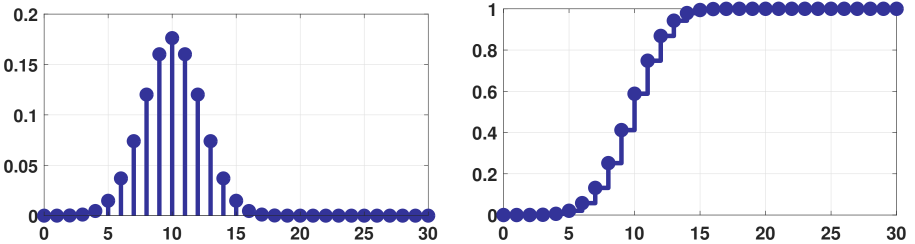


Table: PMF and CDF of a binomial random variable $X \sim \text{Binomial}(n, p)$.

Going back to the binary sensor...

How to model the random variable X = number of 1's observed in 100 measurements?

Questions?