ECE 302: Lecture 3.3 Cumulative Distribution Functions (discrete case)

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3.1 Random variables

3.2 Probability mass functions (PMF)

3.3 Cumulative distribution functions (discrete case)
   - 3.3.1 Generating random numbers
   - 3.3.2 Cumulative distribution functions (CDF)
   - 3.3.3 Properties of CDFs

3.4 Expectation

3.5 Moments and variance

3.6 Bernoulli random variables

3.7 Binomial random variables

3.8 Geometric random variables

3.9 Poisson random variables
Question: How to generate random number from a PMF

$$p_X(k) = [0.1 \ 0.4 \ 0.2 \ 0.3]?$$

Issue: A computer can only generate pre-define distributions, e.g., uniform.
Generating arbitrary random numbers

**Trick:** Compute the **cumulative sum**.

Then what?
The power of cumulative sum

- Why study cumulative sum?
- Relationship to PMF?
- Properties of cumulative sum?
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Cumulative Distribution Function

Definition

The cumulative distribution function (CDF) of a discrete random variable $X$ is

$$F_X(x) \overset{\text{def}}{=} \mathbb{P}[X \leq x] = \sum_{x' \leq x} p_X(x').$$

Interpretation:

- CDF is the “integration” of PMF
- CDF is *well-defined* whereas PMF is not quite
- CDF works for both discrete and continuous random variables
Example 1

Example. Consider a random variable $X$ with PMF

$$p_X(0) = \frac{1}{4}, \quad p_X(1) = \frac{1}{2}, \quad p_X(4) = \frac{1}{4}.$$ 

Find and sketch CDF.
Example. English letters.

(a) PMF $p_X(k)$

(b) CDF $F_X(k)$
### Outline

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  - 3.3.1 Generating random numbers
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  - 3.3.3 Properties of CDFs
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Properties of CDF

1. The CDF is a sequence of increasing

2. $F_X(\infty) =$

3. $F_X(-\infty) =$

4. At positions where $p_X(x) > 0$, there is always a
Properties of CDF

5. The height of each jump is

6. The solid dot is always on the
Proposition

If $X$ is a discrete random variable, then the PMF of $X$ can be obtained from the CDF by

$$p_X(x_k) = F_X(x_k) - F_X(x_{k-1}).$$  

(1)

A simpler version:

$$p_X(k) = F_X(k) - F_X(k - 1).$$  

(2)
Example. If we are given the CDF

\[ F_X(0) = \frac{1}{4}, \quad F_X(1) = \frac{3}{4}, \quad F_X(4) = 1, \]

how do we find the PMF? We know that the PMF will have non-negative values only at \( x = 0, 1, 4 \). For each of these \( x \), we can show that

\[ p_X(0) = F_X(0) - F_X(-\infty) = \frac{1}{4} - 0 = \frac{1}{4}, \]
\[ p_X(1) = F_X(1) - F_X(0) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}, \]
\[ p_X(4) = F_X(4) - F_X(1) = 1 - \frac{3}{4} = \frac{1}{4}. \]
Expressing PMF as delta functions

A PMF can technically be viewed as:

\[
p_X(x) = \sum_{k \in X(\Omega)} p_X(k) \cdot \delta(x - k).
\]  

(3)

For example,

\[
p_X(x) = \frac{1}{4} \delta(x) + \frac{1}{2} \delta(x - 1) + \frac{1}{4} \delta(x - 2).
\]

Then CDF is

\[
F_X(x) = \frac{1}{4} u(x) + \frac{1}{2} u(x - 1) + \frac{1}{4} u(x - 2)
\]

- \(p_X(x)\) is not a function
- \(F_X(x)\) is a function
CDF = cum-sum of PDF
CDF is always defined; some textbooks prefer to define PMF as derivatives of CDF
CDF has several properties
CDF of discrete random variables are stair case functions
CDF can be used to generate random numbers of arbitrary distribution
We will discuss CDF again when we talk about continuous random variables
Questions?