ECE 302: Lecture 3.2 Probability Mass Functions

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3.1 Random variables

3.2 Probability mass functions (PMF)
   3.2.1 Everything you need to know about a PMF
   3.2.2 Histogram
   3.2.3 Properties of a PMF

3.3 Cumulative distribution functions

3.4 Expectation

3.5 Moments and variance

3.6 Bernoulli random variables

3.7 Binomial random variables

3.8 Geometric random variables

3.9 Poisson random variables
Random variables are functions that translate words to numbers!

- So many numbers
- So many events
- How to systematically describe them?
Probability Mass Function

Definition

The **probability mass function (PMF)** of a random variable \( X \) is a function which specifies the probability of obtaining a number \( X(\xi) = a \). We denote a PMF as

\[
p_X(a) = \mathbb{P}[X = a].
\]

- There are two functions here:
  - Function \( X \): the random variable which translates words to numbers
  - Function \( p_X \): the mapping from event \( \{X = a\} \) to a probability

- Difference between \( X \) and \( a \):
  - \( X \) is the **random variable**. Technically it should be \( X(\xi) \)
  - \( a \) is a **state**. So \( X = a \) means \( X \) is taking the state \( a \).

A random variable is random because it has many states!
Example

Flip a coin twice. Define $X$ = number of heads. Then the probability mass function is

\[ p_X(0) = \mathbb{P}[X = 0] = \mathbb{P}\left[\{ "TT" \}\right] = \frac{1}{4}, \]

\[ p_X(1) = \mathbb{P}[X = 1] = \mathbb{P}\left[\{ "TH", "HT" \}\right] = \frac{1}{2}, \]

\[ p_X(2) = \mathbb{P}[X = 2] = \mathbb{P}\left[\{ "HH" \}\right] = \frac{1}{4}. \]
Difference between variable and random variable

Solve an equation $2X + a = 0$.

- If $a$ is fixed, then $X$ is an ordinary variable
- If $a$ has multiple states, and is random, then $X$ is a random variable.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Ordinary Variable</th>
<th>Random Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2X + 1 = 0$</td>
<td>$X = -\frac{1}{2}$</td>
<td>$X = +\frac{1}{2}$ or $X = 0$ or $X = -\frac{1}{2}$</td>
</tr>
<tr>
<td>$2X + 0 = 0$ or $2X - 1 = 0$</td>
<td></td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Solution</th>
<th>Ordinary Variable</th>
<th>Random Variable</th>
</tr>
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<td>$X = -\frac{1}{2}$</td>
<td></td>
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</tbody>
</table>

| Random? | No | Yes |
| No. of states | 1 (deterministic) | 3 |
| PMF | Not applicable | $p_X(x) = \frac{1}{3}$, for $x = \frac{1}{2}, 0, -\frac{1}{2}$ |
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Histogram

We all know what histograms are.

Figure: The frequency of the 26 English letters. Data source: Wikipedia.

- A histogram contains state, and their probability.
**Question:** What is the difference between a PMF and a histogram?

**Answer:** PMF is the ideal histogram!

![Histogram and PMF](image-url)

**Figure:** Histogram and PMF, when throwing a fair dice $N$ times. As $N$ increases, the histograms are becoming more like the PMF.
Why border to study PMF?

- Ideal vs empirical. You always want something ideal.
- Modeling the data. Histogram is non-parametric.
- Generator. Synthesize data, and test whether your algorithm will give what you want.
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Properties of PMF

**Theorem**

A PMF should satisfy the condition that

$$\sum_{x \in X(\Omega)} p_X(x) = 1.$$  

(1)

**Proof.**

$$\sum_{x \in X(\Omega)} P[X = x] = \sum_{x \in X(\Omega)} P[\{\xi \in \Omega | X(\xi) = x\}]$$

$$= P \left[ \bigcup_{\xi \in \Omega} \{\xi \in \Omega | X(\xi) = x\} \right] = P[\Omega] = 1. \quad \square$$
Examples

Example 1. Let $X$ be a random variable with PMF

$$p_X(k) = c \left( \frac{1}{2} \right)^k, \quad k = 1, 2, \ldots$$

Find $c$. 

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![PMF of $X$ for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.]
Example 2. Let $X$ be a random variable with PMF

$$p_X(k) = c \sin \left( \frac{\pi}{2} k \right),$$

for $k = 0, 1, 2, \ldots$. Find $c$. 

![Graph of $p_X(k)$ for $k = 0, 1, 2, \ldots$]
Questions?