ECE 302: Lecture 3.2 Probability Mass Functions

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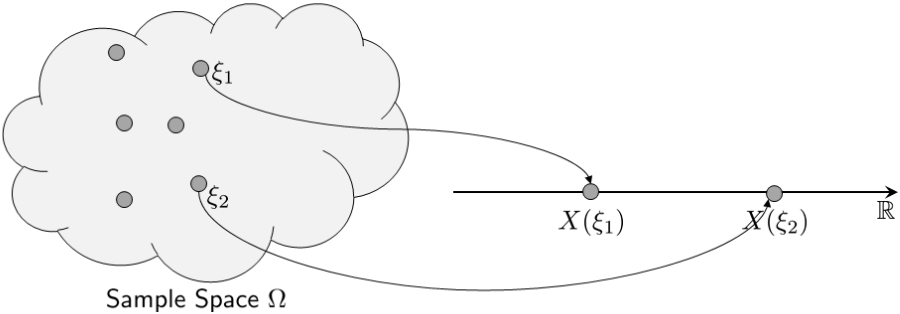
Outline

Everything you need to know about random variables

Random variables are functions that translate words to numbers!



- So many numbers
- So many events
- How to systematically describe them?

Probability Mass Function

Definition

The **probability mass function (PMF)** of a random variable X is a function which specifies the probability of obtaining a number $X(\xi) = a$. We denote a PMF as

$$p_X(a) = \mathbb{P}[X = a].$$

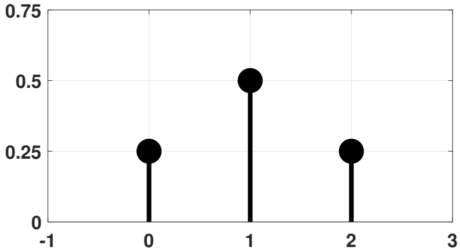
- There are two functions here:
  - Function X: the random variable which translates words to numbers
  - Function p_X: the mapping from event $\{X = a\}$ to a probability
- Difference between X and a:
  - X is the **random variable**. Technically it should be $X(\xi)$
  - a is a **state**. So $X = a$ means X is taking the state a.

A random variable is random because it has many states!

Example

Flip a coin twice. Define X = number of heads. Then the probability mass function is

$$p_X(0) = \mathbb{P}[X = 0] = \mathbb{P}[\{\text{"TT"}\}] = \frac{1}{4},$$
$$p_X(1) = \mathbb{P}[X = 1] = \mathbb{P}[\{\text{"TH"}, \text{"HT"}\}] = \frac{1}{2},$$
$$p_X(2) = \mathbb{P}[X = 2] = \mathbb{P}[\{\text{"HH"}\}] = \frac{1}{4}.$$



Difference between variable and random variable

Solve an equation $2X + a = 0$.

- If a is fixed, then X is an ordinary variable
- If a has multiple states, and is random, then X is a random variable.

	Ordinary Variable	Random Variable
Equation	$2X + 1 = 0$	$2X + 1 = 0$ or $2X + 0 = 0$ or $2X - 1 = 0$
Solution	$X = -\frac{1}{2}$	$X = +\frac{1}{2}$ or $X = 0$ or $X = -\frac{1}{2}$
Random?	No	Yes
No. of states	1 (deterministic)	3
PMF	Not applicable	$p_X(x) = \frac{1}{3}$, for $x = \frac{1}{2}, 0, -\frac{1}{2}$

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Histogram

We all know what histograms are.

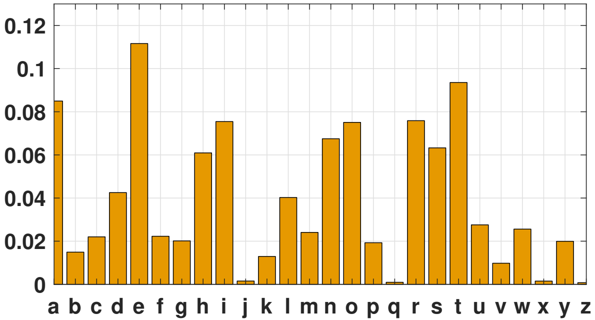
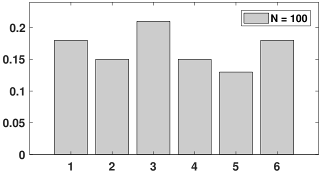


Figure: The frequency of the 26 English letters. Data source: Wikipedia.

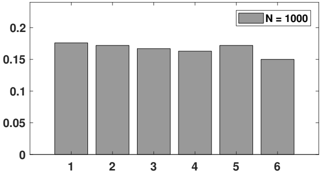
- A histogram contains **state**, and their **probability**.

Histogram VS PMF

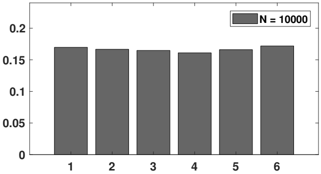
Question: What is the difference between a PMF and a histogram?
Answer: PMF is the **ideal** histogram!



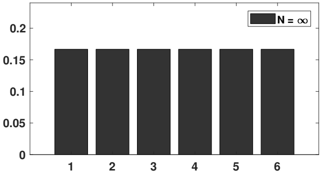
(a) $N = 100$



(b) $N = 1000$



(c) $N = 10000$

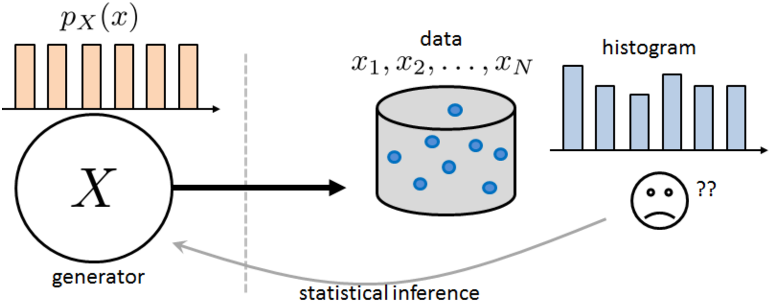


(d) PMF

Figure: Histogram and PMF, when throwing a fair dice N times. As N increases, the histograms are becoming more like the PMF.

Why border to study PMF?

- Ideal vs empirical. You always want something ideal.
- Modeling the data. Histogram is non-parametric.
- Generator. Synthesize data, and test whether your algorithm will give what you want.



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Properties of PMF

Theorem

A PMF should satisfy the condition that

$$\sum_{x \in X(\Omega)} p_X(x) = 1. \tag{1}$$

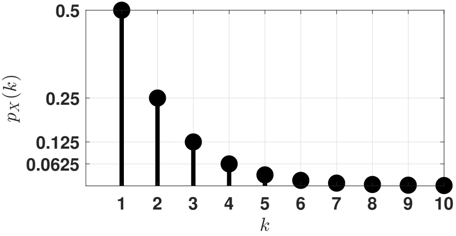
Proof.

$$\begin{aligned} \sum_{x \in X(\Omega)} \mathbb{P}[X = x] &= \sum_{x \in X(\Omega)} \mathbb{P}\left[\{\xi \in \Omega \,|\, X(\xi) = x\}\right] \\ &= \mathbb{P}\left[\bigcup_{\xi \in \Omega} \{\xi \in \Omega \,|\, X(\xi) = x\}\right] = \mathbb{P}[\Omega] = 1. \qquad \square \end{aligned}$$

Example 1. Let X be a random variable with PMF

$$p_X(k) = c\left(\frac{1}{2}\right)^k, \qquad k = 1, 2, \ldots$$

Find c.

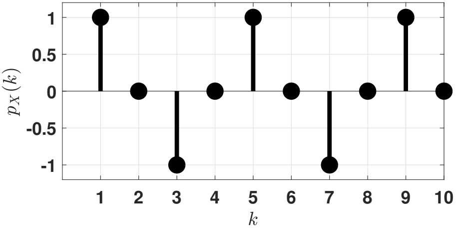


Examples

Example 2. Let X be a random variable with PMF

$$p_X(k) = c \sin\left(\frac{\pi}{2} k\right),$$

for $k = 0, 1, 2, \ldots$. Find c.



Questions?