

ECE 302: Lecture 2.6 Bayes Theorem and Total Probability

Prof Stanley Chan

School of Electrical and Computer Engineering
Purdue University



Outline

- 2.1 Set theory
- 2.2 Probability space
- 2.3 Axioms of probability
- 2.4 Conditional probability
- 2.5 Independence
- 2.6 Bayes theorem
 - 2.6.1 Prisoner's Dilemma
 - 2.6.2 Bayes Theorem and Law of Total Probability
 - 2.6.3 Examples

Prisoner's Dilemma

- Three Prisoners: A, B, C. The King decides to release 2 and sentence 1.
- You were A.
- Your chance of release is $2/3$.
- Suppose you know the guard well. You can ask him about which of B or C will be released.
- But if you find out B (or C) is released, your chance becomes $1/2$.
- How come!!

Conditional Probability

- $X_A =$ sentence A, $X_B =$ sentence B, $X_C =$ sentence C.
- $G_B =$ guard says B is released.

Let us find the probabilities:

- $\mathbb{P}[X_A]$, $\mathbb{P}[X_B]$, $\mathbb{P}[X_C]$

Conditional Probability

- $X_A =$ sentence A, $X_B =$ sentence B, $X_C =$ sentence C.
- $G_B =$ guard says B is released.

Let us find the probabilities:

- $\mathbb{P}[G_B | X_A], \mathbb{P}[G_B | X_B], \mathbb{P}[G_B | X_C]$

Stuck...

$$\mathbb{P}[X_A | G_B] =$$

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Bayes Theorem

Theorem (**Bayes Theorem**)

For any two events A and B such that $\mathbb{P}[A] > 0$ and $\mathbb{P}[B] > 0$, it holds that

$$\mathbb{P}[A | B] = \frac{\mathbb{P}[B | A] \mathbb{P}[A]}{\mathbb{P}[B]}.$$

Law of Total Probability

Theorem (Law of Total Probability)

Let $\{A_1, A_2, \dots, A_n\}$ be a partition of Ω , i.e., A_1, \dots, A_n are disjoint and $\Omega = A_1 \cup A_2 \cup \dots \cup A_n$. Then, for any $B \subseteq \Omega$,

$$\mathbb{P}[B] = \sum_{i=1}^n \mathbb{P}[B | A_i] \mathbb{P}[A_i].$$

Law of Total Probability

continue...

Law of Total Probability

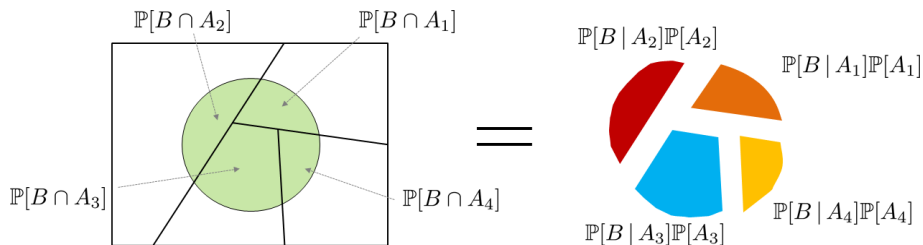


Figure: Law of total probability decomposes the probability $\mathbb{P}[B]$ into multiple conditional probabilities $\mathbb{P}[B | A_i]$. The probability of obtaining each $\mathbb{P}[B | A_i]$ is $\mathbb{P}[A_i]$.

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Example 1: Tennis Tournament

Example. Consider a tennis tournament. Your probability of winning the game is

0.3 against $\frac{1}{2}$ of the players (Event A).

0.4 against $\frac{1}{4}$ of the players (Event B).

0.5 against $\frac{1}{4}$ of the players (Event C).

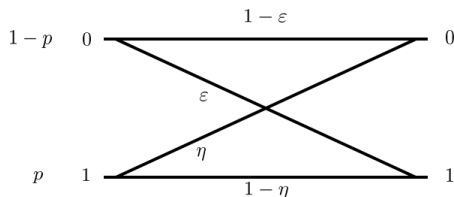
What is the probability of winning the game?

Example 1

continue...

Example 2: Communication Channel

Example. Consider a communication channel shown below. The probability of sending a 1 is p and the probability of sending a 0 is $1 - p$. Given that 1 is sent, the probability of receiving 1 is $1 - \eta$. Given that 0 is sent, the probability of receiving 0 is $1 - \varepsilon$.



- Find $\mathbb{P}[\text{Receive 1}]$
- Find $\mathbb{P}[\text{Send 1}]$
- Find $\mathbb{P}[\text{Receive 1} \mid \text{Send 1}]$
- Find $\mathbb{P}[\text{Send 1} \mid \text{Receive 1}]$

Example 2

continue...

Example 2

continue...

Prisoner's Dilemma (continue)

$$\mathbb{P}[X_A | G_B] =$$

Questions?