ECE 302: Lecture 2.2 Probability Space

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Outline

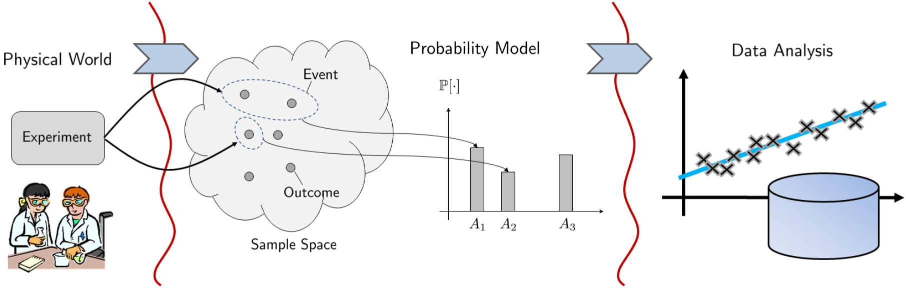
What is Probability?

- It is a **number**.
- Always between **0 and 1**.
- Always the probability of **an event**.

Example. The probability of getting a Head when tossing a coin:

$$\mathbb{P}(\text{“H”}) =$$

Three Elements of a Probability Model



1. Sample Space
2. Event
3. Probability Law

Sample Space

Definition (Sample Space)

A **sample space** Ω is the collection of all possible outcomes.

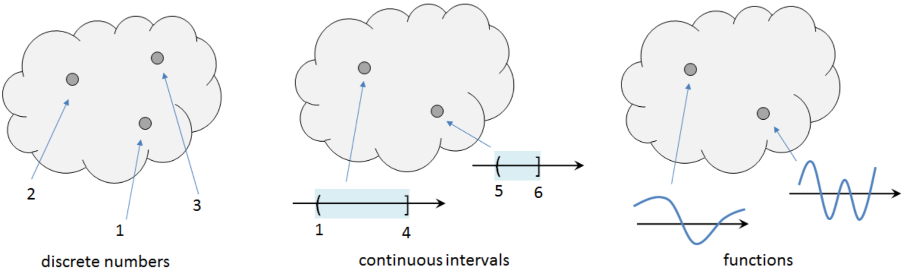
We denote ω as an element in Ω.

Example.

- Coin flip:
  $\Omega =$
- Throw a dice:
  $\Omega =$
- Waiting time for a bus in West Lafayette:
  $\Omega =$

Sample Space

Elements in the sample space can be anything.



Outline

Event

Definition (Event)

An **event** F is a subset in the sample space Ω.

Outcome VS Event:

Example. Throw a dice. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$.

- $F_1 = \{\text{even numbers}\} = \{2, 4, 6\}$.
- $F_2 = \{\text{less than } 3\} = \{1, 2\}$.

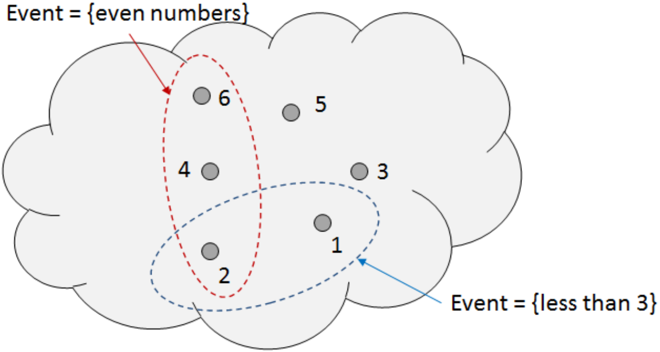
Example. Wait a bus. Let $\Omega = \{0 \leq t \leq 30\}$.

- $F_1 = \{0 \leq t < 10\}$
- $F_2 = \{0 \leq t < 5\} \cup \{20 < t \leq 30\}$.

Event Space

Definition (Event Space)

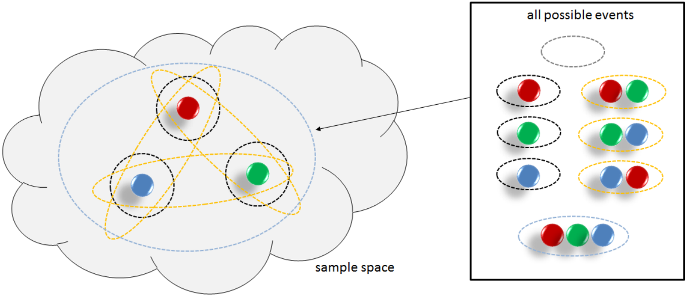
The collection of all events is called the **Event space**, denoted as $\mathcal{F}$



How many events?

Question: If you have n elements in the sample space, how many events can you construct?

Solution:



σ-field (Optional)

The event space is also called the σ-field.

Definition (σ-field)

A σ-**field** $\mathcal{F}$ satisfies the following two properties:

- If $F \in \mathcal{F}$, then $F^c \in \mathcal{F}$
- If $F_1, F_2, \ldots \in \mathcal{F}$, then $F_i \cap F_j \in \mathcal{F}$ and $F_i \cup F_j \in \mathcal{F}$.

Example. $\Omega = \{H, T\}$, the σ-field is $\{\emptyset, H, T, \Omega\}$.

Outline

- 2.1 Set theory
- 2.2 Probability space
  - 2.2.1 Sample space
  - 2.2.2 Event space
  - 2.2.3 Probability law
  - 2.2.4 Measure (Optional)
  - 2.2.5 Probability space
- 2.3 Axioms of probability
- 2.4 Conditional probability
- 2.5 Independence
- 2.6 Bayes theorem

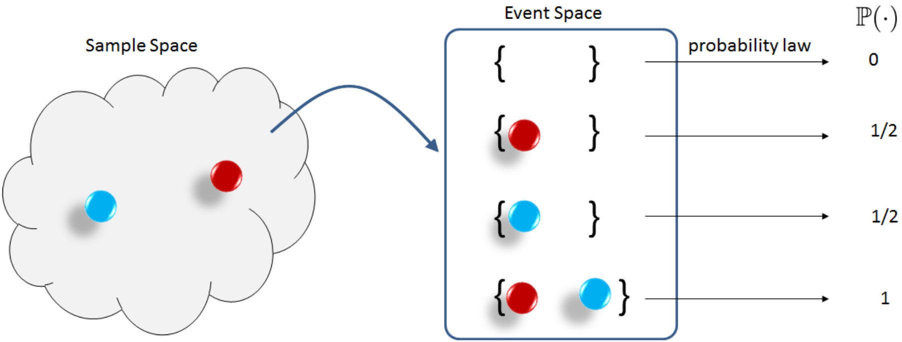
Probability Law

Definition

A **probability law** is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ that maps an event A to a real number in [0, 1].

Example. Consider flipping a coin. The event space $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \Omega\}$. Suggest a probability law that makes sense.

Probability Law



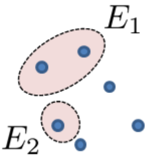
Outline

Probability law as a measure (Optional)

Probability = relative size of a set (w.r.t. the sample space).

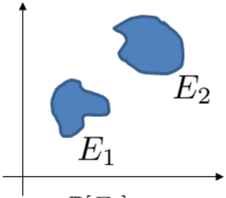
Example:

- Discrete numbers - counting
- 1D intervals - length
- 2D sets - area



$\mathbb{P}[E_1]$ = sum of values

$\mathbb{P}[E_1]$ = length



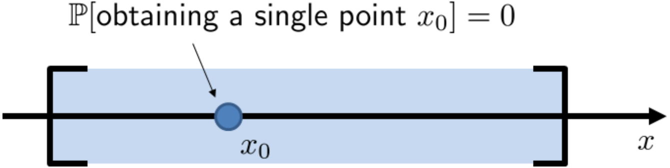
$\mathbb{P}[E_2]$ = area

Measure zero (Optional)

One way to think about probability is to define it as

$$\mathbb{P}[E] = \frac{\text{Size of } E}{\text{Size of } \Omega}$$

Therefore, an isolated point in an interval has zero probability.



Examples of measure zero sets (Optional)

Example 1. Let $\Omega = [0, 1]$. Then the set $\{0.5\}$ has measure zero.

Example 2. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$. Then the set $\{1\}$ has a probability of $1/6$.

Example 3. For any intervals, $\mathbb{P}[[a, b]] = \mathbb{P}[(a, b)]$ because the two end points have measure zero: $\mathbb{P}[\{a\}] = \mathbb{P}[\{b\}] = 0$.

Definition

An event $A \in \mathbb{R}$ is said to hold **almost surely (a.s.)** if

$$\mathbb{P}[A] = 1,$$

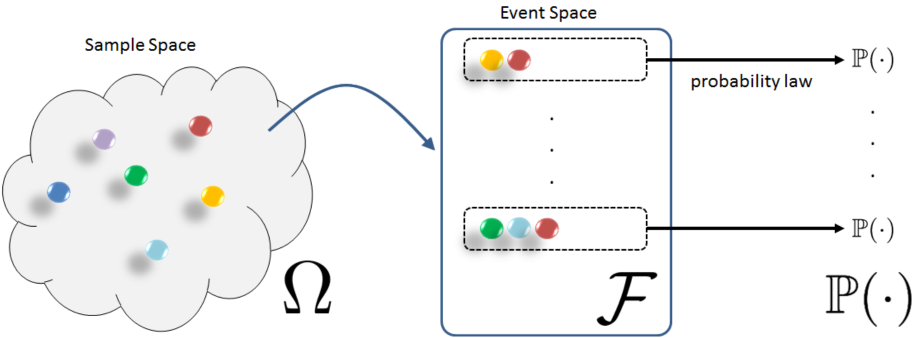
except for all measure-zero sets in $\mathbb{R}$.

Outline

Probability Space

A probability space consists of a triplet:

$$(\Omega, \mathcal{F}, \mathbb{P}) \tag{1}$$



Questions?