ECE 302: Lecture 2.1 Set Theory

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Outline

Set

Definition

A **set** is a collection of objects. We denote $A = \{\xi_1, \xi_2, \ldots, \xi_n\}$ as a set, and ξ_i be the i-th element in the set.

Notation:

- $\xi \in A$: An object ξ is in set A.
- $\xi \notin A$: An object ξ is not in set A.

Finite, Countable, Uncountable.

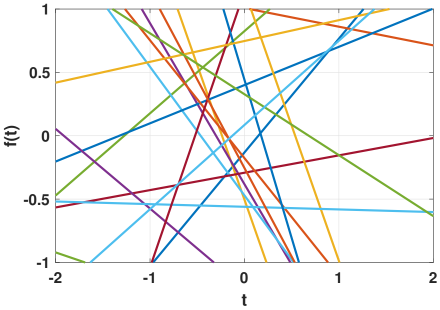
- Finite: $A = \{0, 1\}$
- Countable: $A = \{2, 4, 6, 8, \ldots\}$
- Uncountable: $A = \{x \mid 0 < x < 1\}$

Open and Closed Intervals.

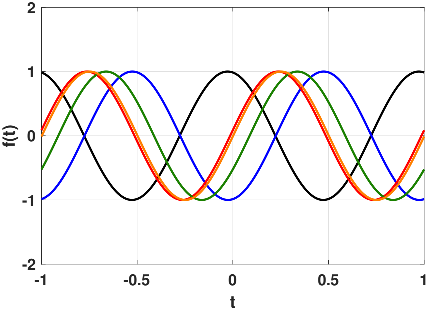
- $(a, b) = \{x \mid a < x < b\}$
- $[a, b] = \{x \mid a \leq x \leq b\}$

Set of functions

A set can contain functions!



(a) Set of straight lines



(b) Set of cosines

Subset

Definition

A **subset** of A is sub-collection of objects in A. That is, $B \subseteq A$ if for any $\xi \in B$, this ξ is also in A.

Proper subset: $B \subset A$.

Example. If $A = \{1, 2, 3, 4, 5, 6\}$, then $B = \{1, 2, \}$ is a proper subset of A.
Example. If $A = \{1, 2, 3\}$, then the set $B = \{1, 2, 3\}$ is an improper subset of A.

Theorem

If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Proof. Suppose $A \subset B$ (which means $A \neq B$. You may also consider $B \subset A$). Then there exist $x \in B$ but $\notin A$. But $B \subseteq A$ requires all $x \in B$ also $\in A$. So we reach a contradiction. The only way to resolve the contradiction is to make $A = B$.

Empty and Universal Set

Definition (Empty Set)

A set is **empty** if it contains no element. We denote an empty set as $\emptyset$.

An empty set is a subset of any set.

Definition

The **universal set** is the set containing all elements. We denote a universal set as Ω.

Any set is a subset of Ω, including Ω itself.

Outline

Finite Union

Definition (Finite Union)

The **finite union** of two sets A and B contains all elements in A **or** in B. That is,

$$A \cup B = \{\xi \mid \xi \in A \text{ or } \xi \in B\}.$$

Example. If $A = \{1, 2, 3, 4\}$, $B = \{1, 5, 6\}$, then $A \cup B =$

Example. If $A = \{t \mid 3 < t \leq 4\}$, $B = \{t \mid t \geq 3.5\}$, then $A \cup B =$

Infinite Union

Definition (Infinite Union)

The **infinite union** of $A_1, A_2, \ldots, A_n$ is denoted as

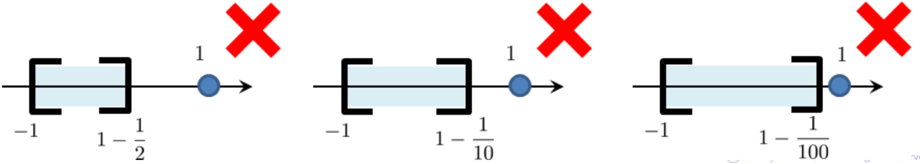
$$A \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} A_n.$$

It holds that $x \in A$ if x is in **at least one** of $A_1, A_2, \ldots, A_n$.

Example. If $A_n = \left[0, 1 - \frac{1}{n}\right]$, then

(a) $A = [0, 1]$

(b) $A = [0, 1)$



Finite Intersection

Definition

The **finite intersection** of two sets A and B contains all elements in A **and** in B. That is,

$$A \cap B = \{\xi \mid \xi \in A \text{ and } \xi \in B\}.$$

Example. If $A = \{1, 2, 3, 4\}$, $B = \{1, 5, 6\}$, then $A \cap B =$

Example. If $A = \{t \mid 3 < t \leq 4\}$, $B = \{t \mid t \geq 3.5\}$, then $A \cap B =$

Infinite Intersection

Definition

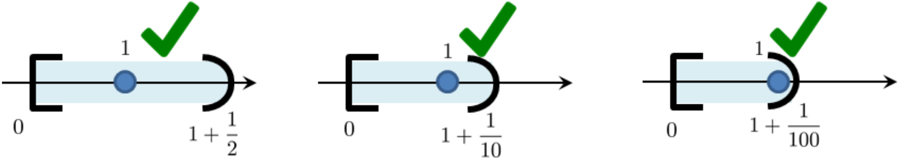
Define the **infinite intersection** of $A_1, A_2, \ldots, A_n$ as

$$A = \bigcap_{n=1}^{\infty} A_n$$

Then, $x \in A$ if x is in **all** of $A_1, A_2, \ldots, A_n$.

Example. If $A_n = [0, 1 + \frac{1}{n})$, then

(a) $A = [0, 1]$
(b) $A = [0, 1)$



Outline

Complement

Definition

The complement of a set A is the set containing all elements in Ω but not in A. That is

$$A^c = \{\xi \mid \xi \in \Omega \text{ and } \xi \notin A\}.$$

Example. Let $A = \{\text{even integers}\}$, $\Omega = \{\text{integers}\}$, then $A^c =$

Difference

Definition

The **difference** $A \backslash B$ is the set containing all elements in A but not in B.

$$A \backslash B = \{\xi \mid \xi \in A \text{ and } \xi \notin B\}.$$

Disjoint

Definition

Two sets A and B are **disjoint** if $A \cap B = \emptyset$. For a collection of sets $\{A_1, A_2, \ldots, A_n\}$, we say that the collection is disjoint if $A_i \cap A_j = \emptyset$.

Partition

Definition

A collection of sets $\{A_1, \ldots, A_n\}$ is a **partition** to the universal set Ω if it satisfies the following conditions:

- (non-overlap) $\{A_1, \ldots, A_n\}$ is disjoint;
- (decompose) $A_1 \cup A_2 \cup \ldots \cup A_n = \Omega$.

Outline

Set Operations

There are four set operations:

- Commutative (Order does not matter).

$$A \cap B = B \cap A$$

$$A \cup B = B \cup A$$

- Associative (How to do multiple union and intersection)

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Set Operations

- Distributive (How to mix union and intersection)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- De Morgan's Law (How to complement over intersection and union)

$$(A \cap B)^c = A^c \cup B^c$$

$$(A \cup B)^c = A^c \cap B^c$$

Questions?