

ECE 302: Lecture 1.4 Linear Algebra (Optional)

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Outline

- 1.1 Infinite Series
 - 1.1.1. Geometric Series
 - 1.1.2. Binomial Series
- 1.2 Approximations
 - 1.2.1. Taylor Approximation
 - 1.2.2. Exponential Series
 - 1.2.3. Logarithmic Approximation
- 1.3 Integration
 - 1.3.1. Odd and Even Functions
 - 1.3.2. Fundamental Theorem of Calculus
- 1.4 Linear Algebra (Optional)
 - 1.4.1. Inner Products (Optional)
 - 1.4.2. Matrix Calculus (Optional)
 - 1.4.3. Matrix Inversion (Optional)
- 1.5 Combinatorics
 - 1.5.1. Permutation
 - 1.5.2. Combination

Basic Notation

- Vector: $\mathbf{x} \in \mathbb{R}^n$
- Matrix: $\mathbf{A} \in \mathbb{R}^{m \times n}$; Entries are a_{ij} or $[\mathbf{A}]_{ij}$.
- Transpose:

$$\mathbf{A} = \begin{bmatrix} | & | & \dots & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \\ | & | & \dots & | \end{bmatrix}, \quad \text{and} \quad \mathbf{A}^T = \begin{bmatrix} \text{---} & \mathbf{a}_1^T & \text{---} \\ \text{---} & \mathbf{a}_2^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a}_n^T & \text{---} \end{bmatrix}.$$

- Column: \mathbf{a}_i is the i -th column of \mathbf{A}
- Identity matrix \mathbf{I}
- All-one vector $\mathbf{1}$ and all-zero vector $\mathbf{0}$
- Standard basis \mathbf{e}_j .

Inner Product

Definition

Let $\mathbf{x} = [x_1, x_2, \dots, x_N]^T$ and $\mathbf{y} = [y_1, y_2, \dots, y_N]^T$ be two vectors. The **inner product** $\mathbf{x}^T \mathbf{y}$ is

Example. Let $\mathbf{x} = [x_1, x_2]^T$. The inner product $\mathbf{x}^T \mathbf{x} =$

Inner product

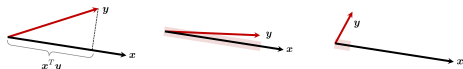


Figure: Geometric interpretation of inner product: We project one vector onto the other vector. The projected distance is the inner product.

Weighted Inner Product

Example.

Let $\mathbf{x} = [x_1, x_2]^T$, $\boldsymbol{\mu} = [\mu_1, \mu_2]$ and $\mathbf{C} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. The product $(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{C}(\mathbf{x} - \boldsymbol{\mu})$ is

The ℓ_2 -norm

Also called the **Euclidean norm**:

Definition

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2}. \quad (1)$$

- The set $\Omega = \{\mathbf{x} \mid \|\mathbf{x}\|_2 \leq r\}$ defines a circle:

$$\Omega = \{\mathbf{x} \mid \|\mathbf{x}\|_2 \leq r\} = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq r^2\}.$$

- $f(\mathbf{x}) = \|\mathbf{x}\|_2$ is not the same as $f(\mathbf{x}) = \|\mathbf{x}\|_2^2$.
- Triangle inequality holds:

$$\|\mathbf{x} + \mathbf{y}\|_2 \leq \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2.$$

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Matrix Calculus

Definition

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a scalar field. The gradient of f with respect to $\mathbf{x} \in \mathbb{R}^n$ is defined as

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix}. \quad (2)$$

Example 1. $f(\mathbf{x}) = \mathbf{a}^T \mathbf{x}$. In this case, the gradient is

$$\nabla_{\mathbf{x}} (\mathbf{a}^T \mathbf{x}) = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} \sum_{j=1}^n a_j x_j \\ \vdots \\ \frac{\partial}{\partial x_n} \sum_{j=1}^n a_j x_j \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \mathbf{a}. \quad (3)$$

More Examples

Example 2. $f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$. Then,

$$\begin{aligned} \nabla_{\mathbf{x}} (\mathbf{x}^T \mathbf{A} \mathbf{x}) &= \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} \sum_{i,j=1}^n a_{ij} x_i x_j \\ \vdots \\ \frac{\partial}{\partial x_n} \sum_{i,j=1}^n a_{ij} x_i x_j \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=1}^n a_{1,j} x_j \\ \vdots \\ \sum_{j=1}^n a_{n,j} x_j \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^n a_{i,1} x_i \\ \vdots \\ \sum_{i=1}^n a_{i,n} x_i \end{bmatrix} = \mathbf{A} \mathbf{x} + \mathbf{A}^T \mathbf{x} \end{aligned}$$

If \mathbf{A} is symmetric so that $\mathbf{A} = \mathbf{A}^T$ then $\nabla_{\mathbf{x}} f(\mathbf{x}) = 2\mathbf{A} \mathbf{x}$

More Examples

Example 3. $f(\mathbf{x}) = \|\mathbf{Ax} - \mathbf{y}\|^2$. The gradient is

$$\begin{aligned}\nabla_{\mathbf{x}}\left(\|\mathbf{Ax} - \mathbf{y}\|^2\right) &= \nabla_{\mathbf{x}}\left(\mathbf{x}^T \mathbf{A}^T \mathbf{Ax} - 2\mathbf{y}^T \mathbf{Ax} + \mathbf{y}^T \mathbf{y}\right) \\ &= \nabla_{\mathbf{x}}\left(\mathbf{x}^T \mathbf{A}^T \mathbf{Ax}\right) - 2\nabla_{\mathbf{x}}\left(\mathbf{y}^T \mathbf{Ax}\right) + \nabla_{\mathbf{x}}\left(\mathbf{y}^T \mathbf{y}\right) \\ &= 2\mathbf{A}^T \mathbf{Ax} - 2\mathbf{A}^T \mathbf{y} + 0 = 2\mathbf{A}^T (\mathbf{Ax} - \mathbf{y}).\end{aligned}$$

Definition

The Hessian of f with respect to $\mathbf{x} \in \mathbb{R}^n$ is defined as

$$\nabla_{\mathbf{x}}^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_n^2} \end{bmatrix}. \quad (4)$$

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Determinant and Inverse

Definition

Determinant Let $\Sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of Σ is

Definition (Inverse)

Let $\Sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse of Σ is

Questions?