ECE 302: Lecture 1.2 Approximation

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Outline

Taylor Approximation

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with infinite derivatives. Let $a \in \mathbb{R}$ be a fixed constant. The Taylor approximation of f at $x = a$ is

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \end{aligned}$$

where $f^{(n)}$ denotes the n-th order derivative of f.

Example 1

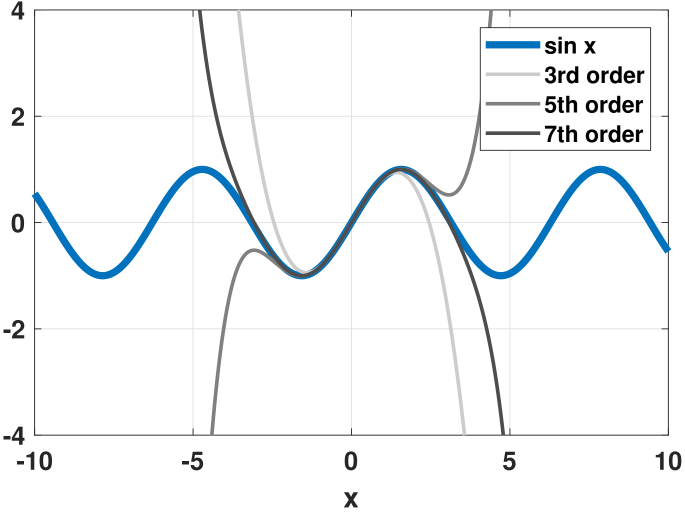
Find Taylor approximation of $f(x) = \sin x$ at $x = 0$.

Solution. The Taylor approximation at $x = 0$ is

$$
\begin{aligned}
f(x) &\approx f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 \\
&= \sin(0) + (\cos 0)(x-0) - \frac{\sin(0)}{2!}(x-0)^2 - \frac{\cos(0)}{3!}(x-0)^3 \\
&= x - \frac{x^3}{6}.
\end{aligned}
$$

We can expand further to higher orders, which yields

$$
f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots
$$

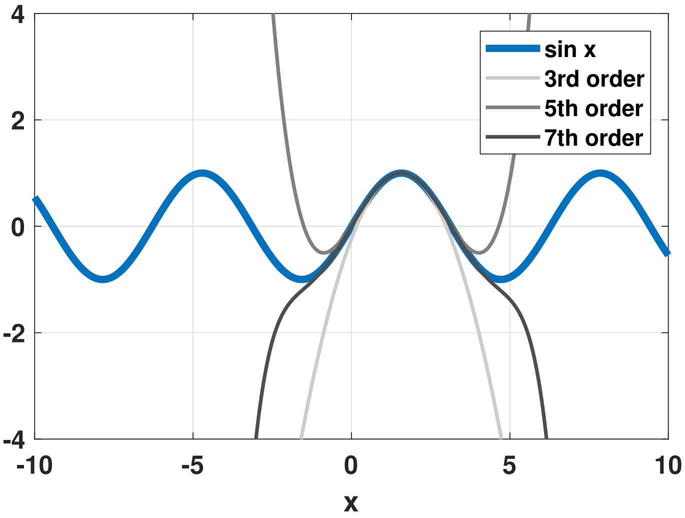

4
2
0
-2
-4
-10
-5
0
5
10
x
sin x
3rd order
5th order
7th order

Example 2

Find Taylor approximation of $f(x) = \sin x$ at $x = 1/2$.

Solution. Taylor approximation at $x = \pi/2$ for $f(x) = \sin x$ is

$$
\begin{aligned}
f(x) &= \sin\frac{\pi}{2} + \cos\frac{\pi}{2}\left(x - \frac{\pi}{2}\right) - \frac{\sin\frac{\pi}{2}}{2!}\left(x - \frac{\pi}{2}\right)^2 - \frac{\cos\frac{\pi}{2}}{3!}\left(x - \frac{\pi}{2}\right)^3 \\
&= 1 - \frac{1}{4}\left(x - \frac{\pi}{2}\right)^2.
\end{aligned}
$$



sin x
3rd order
5th order
7th order
4
2
0
-2
-4
-10
-5
0
5
10
x

Outline

Exponential Series

Theorem

Let x be any real number. Then,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{1}$$

Proof. Let $f(x) = e^x$ for any x. Then, Taylor approximation at $x = 0$ is

$$\begin{aligned} f(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \ldots \\ &= e^0 + e^0(x-0) + \frac{e^0}{2!}(x-0)^2 + \ldots \\ &= 1 + x + \frac{x^2}{2} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

This result will be used in **Poisson** random variables.

Example 1

Evaluate the sum $\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}$.

Example 2

Substitute $x = j\theta$ where $j = \sqrt{-1}$. Find the Taylor approximation of sine and cosine.

Outline

Logarithmic Approximation

Theorem

Let $0 < x < 1$ *be a constant. Then,*

$$\log(1+x) = x - x^2 + \mathcal{O}(x^3). \tag{2}$$

Proof. Let $f(x) = \log(1+x)$. Then, the derivatives of f are

$$f'(x) = \frac{1}{(1+x)}, \quad \text{and} \quad f''(x) = -\frac{1}{(1+x)^2}.$$

Taylor approximation at $x = 0$ gives

$$\begin{aligned} f(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \mathcal{O}(x^3) \\ &= x - x^2 + \mathcal{O}(x^3). \end{aligned}$$

This result will be used in **Central Limit Theorem**.

Example

Show that

$$\lim_{N\to\infty}\left(1+\frac{s^2}{2N}\right)^N = e^{s^2/2}. \tag{3}$$

Questions?