ECE 302: Lecture 1.2 Approximation

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Outline

- 1.1 Infinite Series
  - 1.1.1. Geometric Series
  - 1.1.2. Binomial Series
- 1.2 Approximations
  - 1.2.1. Taylor Approximation
  - 1.2.2. Exponential Series
  - 1.2.3. Logarithmic Approximation
- 1.3 Integration
  - 1.3.1. Odd and Even Functions
  - 1.3.2. Fundamental Theorem of Calculus
- 1.4 Linear Algebra (Optional)
  - 1.4.1. Inner Products (Optional)
  - 1.4.2. Matrix Calculus (Optional)
  - 1.4.3. Matrix Inversion (Optional)
- 1.5 Combinatorics
  - 1.5.1. Permutation
  - 1.5.2. Combination
Taylor Approximation

**Definition**

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with infinite derivatives. Let $a \in \mathbb{R}$ be a fixed constant. The Taylor approximation of $f$ at $x = a$ is

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n,$$

where $f^{(n)}$ denotes the $n$-th order derivative of $f$. 
Example 1

Find Taylor approximation of \( f(x) = \sin x \) at \( x = 0 \).

**Solution.** The Taylor approximation at \( x = 0 \) is

\[
f(x) \approx f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \frac{f'''(0)}{3!}(x - 0)^3
\]

\[
= \sin(0) + (\cos 0)(x - 0) - \frac{\sin(0)}{2!}(x - 0)^2 - \frac{\cos(0)}{3!}(x - 0)^3
\]

\[
= x - \frac{x^3}{6}.
\]

We can expand further to higher orders, which yields

\[
f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]
Example 2

Find Taylor approximation of $f(x) = \sin x$ at $x = 1/2$.

**Solution.** Taylor approximation at $x = \pi/2$ for $f(x) = \sin x$ is

$$f(x) = \sin \frac{\pi}{2} + \cos \frac{\pi}{2} \left( x - \frac{\pi}{2} \right) - \frac{\sin \frac{\pi}{2}}{2!} \left( x - \frac{\pi}{2} \right)^2 - \frac{\cos \frac{\pi}{2}}{3!} \left( x - \frac{\pi}{2} \right)^3$$

$$= 1 - \frac{1}{4} \left( x - \frac{\pi}{2} \right)^2 .$$
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Exponential Series

Theorem

Let $x$ be any real number. Then,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$  \hspace{1cm} (1)

Proof. Let $f(x) = e^x$ for any $x$. Then, Taylor approximation at $x = 0$ is

$$f(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \ldots$$

$$= e^0 + e^0(x - 0) + \frac{e^0}{2!}(x - 0)^2 + \ldots$$

$$= 1 + x + \frac{x^2}{2} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$  

This result will be used in Poisson random variables.
Example 1

Evaluate the sum \( \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \).
Example 2

Substitute $x = j\theta$ where $j = \sqrt{-1}$. Find the Taylor approximation of sine and cosine.
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Logarithmic Approximation

**Theorem**

Let $0 < x < 1$ be a constant. Then,

\[
\log(1 + x) = x - x^2 + O(x^3).
\]

(2)

**Proof.** Let $f(x) = \log(1 + x)$. Then, the derivatives of $f$ are

\[
f'(x) = \frac{1}{1 + x}, \quad \text{and} \quad f''(x) = -\frac{1}{(1 + x)^2}.
\]

Taylor approximation at $x = 0$ gives

\[
f(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)}{2}(x - 0)^2 + O(x^3)
\]

\[= x - x^2 + O(x^3).
\]

This result will be used in Central Limit Theorem.
Example

Show that

$$\lim_{N \to \infty} \left(1 + \frac{s^2}{2N}\right)^N = e^{s^2/2}. \quad (3)$$
Questions?