ECE 302: Lecture 10.6 Power Spectral Density

Prof Stanley Chan

School of Electrical and Computer Engineering
Purdue University

Wide Sense Stationary Processes

Definition

A random process $X(t)$ is **wide sense stationary (W.S.S.)** if:

1. $\mu_X(t) = \text{constant}, \text{ for all } t,$
2. $R_X(t_1, t_2) = R_X(t_1 - t_2) \text{ for all } t_1, t_2.$

Remark 1: WSS processes can also be defined using the autocovariance function

$$C_X(t_1, t_2) = C_X(t_1 - t_2).$$

Remark 2: Because a WSS is completely characterized by the difference $t_1 - t_2$, there is no need to keep track of the absolute indices t_1 and t_2. We can rewrite the autocorrelation function as

$$R_X(\tau) = \mathbb{E}[X(t + \tau)X(t)]. \tag{1}$$

Power of a Random Process

Consider a random process $X(t)$.

Random realization of power: The power within a period $[-T, T]$ is

$$\widehat{P}_X = \frac{1}{2T}\int_{-T}^{T} |X(t)|^2 dt.$$

- Since $X(t)$ is random, the power $\widehat{P}_X$ is also random.
- T is a finite period of time which does not capture the entire process.

Power of a random process:

$$P_X \overset{\text{def}}{=} \mathbb{E}\left[\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |X(t)|^2 dt\right]. \tag{2}$$

Power Spectral Density

Definition

The **power spectral density** (PSD) of a W.S.S. process is defined as

$$S_X(\omega) = \lim_{T\to\infty} \frac{\mathbb{E}\left[|\widetilde{X}_T(\omega)|^2\right]}{2T}, \tag{3}$$

where

$$\widetilde{X}_T(\omega) = \int_{-T}^{T} X(t)e^{-j\omega t}dt \tag{4}$$

is the Fourier transform of $X(t)$ limited to $[-T, T]$.

Einstein-Wiener-Khinchin Theorem

Theorem (Einstein-Wiener-Khinchin Theorem)

The power spectral density $S_X(\omega)$ of a W.S.S. process is

$$\begin{aligned} S_X(\omega) &= \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau \\ &= \mathcal{F}(R_X(\tau)). \end{aligned}$$

Remark: The power spectral density is defined for WSS processes. If the process is not WSS, then R_X will be a 2D function instead of a 1D function in τ. So we cannot take Fourier transform in τ. We will discuss this in details shortly.

Example

Example 1. Let $R_X(\tau) = e^{-2\alpha|\tau|}$. Find $S_X(\omega)$.

Solution. Using the Fourier transform table, we can show that

$$S_X(\omega) = \mathcal{F}\left\{R_X(\tau)\right\} = \frac{4\alpha}{4\alpha^2 + \omega^2}.$$

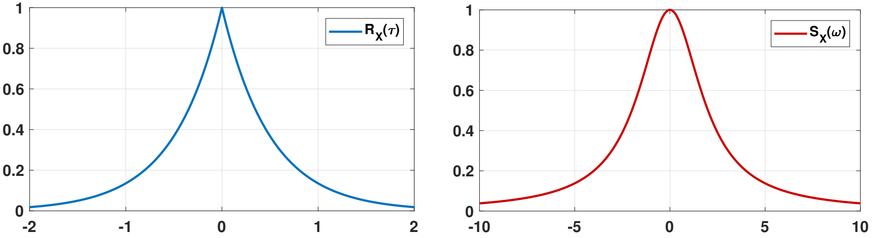


Figure: Example for $R_X(\tau) = e^{-2\alpha|\tau|}$, with $\alpha = 1$.

Example

Example 2. Let $X(t) = a\cos(\omega_0 t + \Theta)$, $\Theta \sim \text{Uniform}[0, 2\pi]$. Find $S_X(\omega)$.

Solution. We know that the autocorrelation function is

$$\begin{aligned} R_X(\tau) &= \frac{a^2}{2}\cos(\omega_0\tau) \\ &= \frac{a^2}{2}\left(\frac{e^{j\omega_0\tau} + e^{-j\omega_0\tau}}{2}\right). \end{aligned}$$

Then, by taking Fourier transform of both sides, we have

$$\begin{aligned} S_X(\omega) &= \frac{a^2}{2}\left[\frac{2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)}{2}\right] \\ &= \frac{\pi a^2}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]. \end{aligned}$$

Example 2

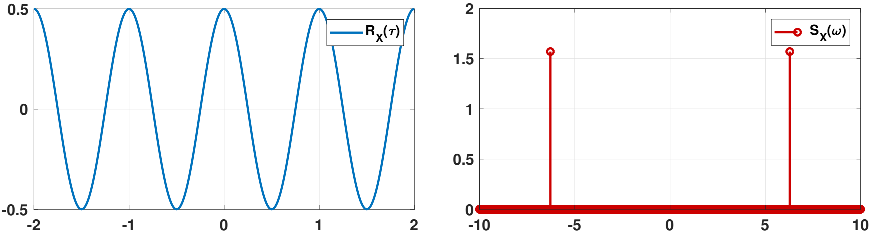


Figure: Example for $R_X(\tau) = \frac{a^2}{2}\cos(\omega_0\tau)$, with $a = 1$ and $\omega_0 = 2\pi$.

Example

Example 3. Let $S_X(\omega) = \frac{N_0}{2}\mathrm{rect}(\frac{\omega}{2W})$. Find $R_X(\tau)$.

Solution. Since $S_X(\omega) = \mathcal{F}(R_X(\tau))$, the inverse holds:

$$R_X(\tau) = \frac{N_0}{2}\frac{W}{\pi}\mathrm{sinc}(W\tau).$$

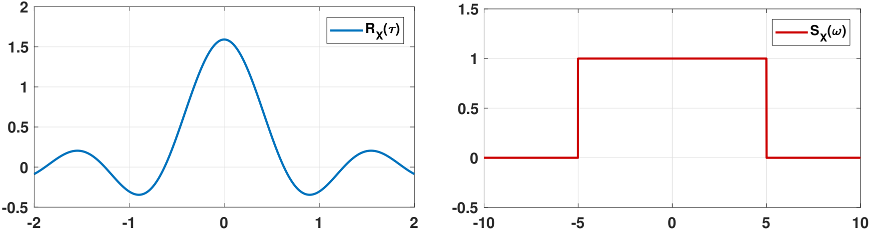


Figure: Example for $S_X(\omega) = \frac{N_0}{2}\mathrm{rect}(\frac{\omega}{2W})$, with $N_0 = 2$ and $W = 5$.

Why study power spectral density?

What is the usage of power spectral density?

- Useful when we pass a random process through some linear operations.
- For example, convolution: running average, or running difference.
- Fourier transform is useful to speed up the computation, and help drawing samples.

Why does power spectral density require WSS?

This has to go with the toeplitz structure of the autocorrelation function.

$$
\begin{aligned}
\boldsymbol{R} &= \begin{bmatrix} R_X[1,1] & R_X[1,2] & \dots & R_X[1,N] \\ R_X[2,1] & R_X[2,2] & \dots & R_X[2,N] \\ \vdots & \vdots & \ddots & \vdots \\ R_X[N,1] & R_X[N,2] & \dots & R_X[N,N] \end{bmatrix} \\
&= \begin{bmatrix} R_X[0] & R_X[1] & \dots & R_X[N-1] \\ R_X[1] & R_X[0] & \dots & R_X[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ R_X[N-1] & R_X[N-1] & \dots & R_X[0] \end{bmatrix},
\end{aligned}
$$

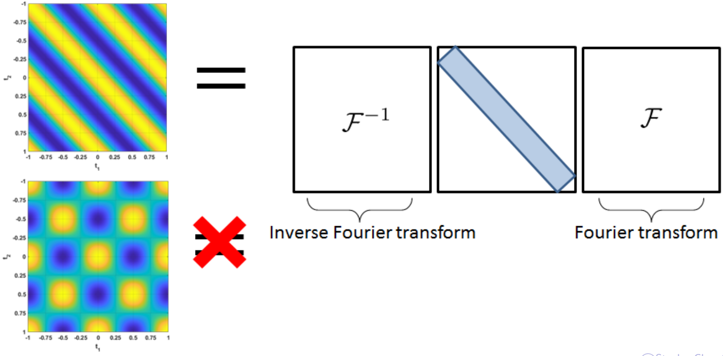
where the second equality holds because $R_X[m,n] = R_X[m-n]$ for WSS processes, and $R_X[k] = R_X[-k]$.

Eigen-decomposition

For a toeplitz matrix $\boldsymbol{R}$, it holds that $\boldsymbol{R}$ can be diagonalized using the Fourier transforms. That is, we can write $\boldsymbol{R}$ as

$$\boldsymbol{R} = \boldsymbol{F}^H \boldsymbol{\Lambda} \boldsymbol{F},$$

where $\boldsymbol{F}$ is the (discrete) Fourier transform matrix, and $\boldsymbol{\Lambda}$ is a diagonal matrix. This can be viewed as the eigen-decomposition of $\boldsymbol{R}$.



Summary

Theorem (Einstein-Wiener-Khinchin Theorem)

The power spectral density $S_X(\omega)$ of a W.S.S. process is

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau$$
$$= \mathcal{F}(R_X(\tau)).$$

Why does power spectral density require WSS?

- Because if a process is WSS, then R_X is toeplitz.
- Fourier transform is the eigenvector of a toeplitz matrix.
- If R_X is not toeplitz, then you cannot diagonalize the correlation matrix.

Questions?