Randomness in $p_{X|Y}(x|y)$

- Fix $Y$ at the state $Y = y$.
- $p_{X|Y}(x|y)$ is the PMF of $X$,
- but under the condition that $Y = y$.
- $p_{X|Y}(x|1)$ and $p_{X|Y}(x|2)$ are two different PMFs,
- although they are both PMFs of $X$.

\[
\sum_{x'} p_{X|Y}(x'|y) = 1
\]
\[
\sum_{y'} p_{X|Y}(x|y') \neq 1
\]

Same result applies to $f_{X|Y}(x|y)$. 
Conditional Expectation

Definition

The conditional expectation of $X$ given $Y = y$ is

$$E[X \mid Y = y] = \sum_x x p_{X \mid Y}(x \mid y)$$

(7)

for the discrete case, and

$$E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X \mid Y}(x \mid y) \, dx$$

(8)

for the continuous case.
Note about Conditional Expectation

- In $\mathbb{E}[X \mid Y = y]$, the expectation is taken over $X$. The randomness is $X$.
- The PDF is $f_{X \mid Y}(x \mid y)$.
- The random variable $Y$ is fixed at $Y = y$.
- The random variable $X$ has been eliminated by the expectation.
- The resulting object $\mathbb{E}[X \mid Y = y]$ is a function of $y$. 
Law of Total Expectation

\textbf{Theorem (Law of Total Expectation)}

\textit{Discrete:}

\[ \mathbb{E}[X] = \sum_y \mathbb{E}[X \mid Y = y] \rho_Y(y). \]  \hfill (9)

\textit{Continuous:}

\[ \mathbb{E}[X] = \int_{-\infty}^{\infty} \mathbb{E}[X \mid Y = y] f_Y(y) \, dy. \]  \hfill (10)
Why Care about Conditional Expectation?

Example 1.

- Two classes of cars. Let $X$ be speed, let $C$ be class.
- When $C = 1$, $X \sim \mathcal{N} (\mu_1, \sigma_1)$. Also, $\mathbb{P}[C = 1] = p$.
- When $C = 2$, $X \sim \mathcal{N} (\mu_2, \sigma_2)$. Also, $\mathbb{P}[C = 2] = 1 - p$.
- If you see a car on the freeway, what is its average speed? \[\mathbb{E}[X]\]
\[
\begin{align*}
\left\{ f_{X|C}(x|1) & = \mathcal{N}(\mu_1, \sigma_1^2) \quad f_C(1) = p \\
\left\{ f_{X|C}(x|2) & = \mathcal{N}(\mu_2, \sigma_2^2) \quad f_C(2) = 1 - p \\
[\mathbb{E}[X]] & = \int_{-\infty}^{\infty} x f_X(x) \, dx \\
& = \int_{-\infty}^{\infty} x \left[ \int_{-\infty}^{\infty} f_{X,C}(x,c) \, dc \right] \, dx \\
& = \int_{-\infty}^{\infty} x \left[ \int_{-\infty}^{\infty} f_{X|C}(x|c) f_C(c) \, dc \right] \, dx \\
\end{align*}
\]

Proof of law of total expectation:

\[
\mathbb{E}[X|C = c] f_C(c) \, dc
\]
\[ E(X|C=1) = \text{conditional expectation of } X|C. \]
\[ E(X|C=2) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx. \]
\[ = \mu_2 \]
\[ = \mu_1 \]

\[ E(X) = \int_{-\infty}^{\infty} E(X|C=c) f_C(c) dc \]
\[ = E(X|C=1) p + E(X|C=2)(1-p). \]
\[ = \mu_1 p + \mu_2 (1-p). \]
Why Care about Conditional Expectation?

The conditional PDFs are:

\[
f_{X|c}(x \mid 1) = \frac{1}{\sqrt{2\pi \sigma_1^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}, \quad f_{X|c}(x \mid 2) = \frac{1}{\sqrt{2\pi \sigma_2^2}} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}}
\]

Therefore, conditioned on \( C \), we have two expectations:

\[
E[X \mid C = 1] = \int_{-\infty}^{\infty} x f_{X|c}(x \mid 1)dx = \mu_1,
\]

\[
E[X \mid C = 2] = \int_{-\infty}^{\infty} x f_{X|c}(x \mid 2)dx = \mu_2.
\]

The overall expectation \( E[X] \) is

\[
\]
Another Example

Example 2.

- You bought a **cheap** guitar tuner.
- The estimated pitch is worse for higher notes.
- Let $Z$ be the true pitch, $Z \sim \mathcal{N}(\mu, \delta^2)$

\[
    f_Z(z) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}
\]

- Let $X$ be the estimated pitch, $X \mid Z \sim \mathcal{N}(Z, Z^2)$

\[
    f_{X\mid Z}(x \mid z) = \frac{1}{\sqrt{2\pi z^2}} e^{-\frac{(x-z)^2}{2z^2}}
\]

- Find $\mathbb{E}[X]$. 

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\[ E[X] = \int x f_X(x) \, dx = \mu. \]

\[ = \int_\infty^\infty x \int_\infty^\infty f_{X,Z}(x,z) \, dz \, dx \]

\[ = \int_\infty^\infty f_{X|Z}(x|z) \, dx \cdot f_Z(z) \, dz \]

\[ = \int_\infty^\infty \frac{1}{\sqrt{2\pi z^2}} \frac{-(x-z)^2}{2z^2} \, dx \]

\[ = \mu \]